1. Two independent sources \( s_1 \) and \( s_2 \), both uniformly distributed on \([-1, 1]\), are mixed either with \( x = A_1 s \) or \( x = A_2 s \), where

\[
A_1 = \begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} / \sqrt{2}.
\]

(a) Draw images about how the linear mappings \( A_1 \) and \( A_2 \) change the coordinate system.

(b) Compute the covariance matrices of the mixtures.

(c) Whiten the mixtures with PCA and visualize the new coordinate systems.

(d) Consider an orthogonal rotation matrix

\[
W = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},
\]

where \( \theta \) is the angle of rotation. Let \( u_1 \) and \( u_2 \) be the components of the whitened mixtures after the rotation \( W \). Check that for any rotation angle \( \theta \), the components \( u_1 \) and \( u_2 \) remain uncorrelated and have constant variance equal to 1. So, decorrelation or variance maximization will not help in finding the correct rotation \( W \).

(e) Plot the kurtosis of \( u_1 \) and \( u_2 \) as functions of \( \theta \). How can the kurtosis be used for finding the rotation \( W \).

(f) Can some other measures be used for finding the rotation?

2. Show that

\[
I(X;Y) = D_{p(X,Y)||p(X)p(Y)},
\]

that is, mutual information is equal to the Kullback-Leibler divergence of the joint distribution from the “corresponding” factored distribution.

3. Show that for a transformation \( y = Wx \) there exists a simple connection between mutual information \( I \) and negentropy \( J \) assuming that the \( Y_i \) are uncorrelated and have unit variance:

\[
I(Y_1, \ldots, Y_n) = C - \sum_{i=1}^{n} J(Y_i),
\]

where \( C \) is a constant. (Notation: \( y_i \) is the value of the random variable \( Y_i \).) What does this connection imply for ICA computation?