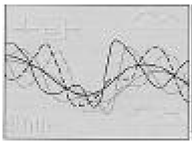
# Sampling



# Introduction

- Under certain conditions, a continuous-time signal can be represented by and recovered from its ***samples*** at points equally spaced in time

→ ***Sampling Theorem***

- The sampling theorem forms a bridge between continuous-time and discrete-time signals and systems
- When the conditions of the sampling theorem are fulfilled continuous-time signals can be processed using discrete-time systems, i.e., using digital signal processing (DSP) methods



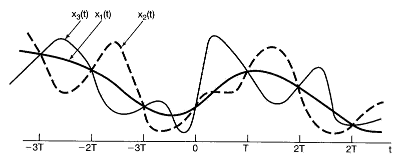
# Advantages of DSP

- DSP systems have several advantages when compared to the CT systems:
  - Flexibility due to (re)programmability
  - Computational accuracy and stability
  - Easy reproducibility
  - Efficient implementation: DSP processors and application specific integrated circuits (ASICs) on VLSI
- DSP systems can be used to implement functions that are not possible in analog signal processing (nonlinear DSP algorithms)
- Digitalization…



# Representation of Continuous-Time Signals by Its Samples: The Sampling Theorem

- In general, in the absence of any additional information, we would not expect that a signal could be uniquely specified by a sequence of equally spaced samples




- An infinite number of signals can generate a given set of samples

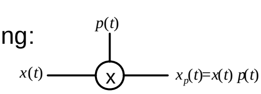


# The Sampling Theorem

If a signal is ***bandlimited*** - i.e., if its Fourier transform is zero outside a finite band of frequencies - and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal, then the samples ***uniquely*** specify the signal, and it can be perfectly reconstructed



## Impulse-Train Sampling:

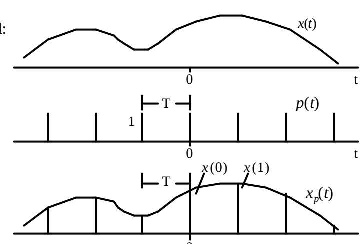



Continuous-time signal:

Impulse train:

$$p(t) = \sum_{n=-\infty}^{\infty} d\,(t-nT)$$

Signal samples:



# Impulse-Train Sampling

- The periodic impulse train $p(t)$ is referred to as the ***sampling function***
- The period $T$ is the ***sampling period***
- The fundamental frequency of $p(t)$, $w_s = 2p/T$ is the ***sampling frequency*** or ***sampling rate***
- In time-domain: $x_p(t) = x(t)p(t)$, where $p(t) = \sum_{n=-\infty}^{\infty} d(t-nT)$
- Multiplying $x(t)$ by a unit impulse, samples the value of the signal at the point at which the impulse is located, i.e.,

$$x(t)d(t-t_0) = x(t_0)d(t-t_0)$$

- Thus, $x_p(t)$ is an impulse train with the amplitudes of the impulses equal to the samples of $x(t)$ at intervals spaced by $T$, i.e.,

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)d(t-nT)$$



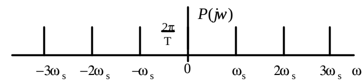

# Impulse-Train Sampling

- From the multiplication property of the convolution theorem, we know that

$$x_p(t) = x(t)p(t) \quad \Leftrightarrow \quad X_p(jw) = \frac{1}{2p}[X(jw) * P(jw)]$$

- The Fourier transform of a periodic impulse train $p(t)$ is also a periodic impulse train in the frequency domain, i.e.,

$$P(jw) = \frac{2p}{T}\sum_{k=-\infty}^{\infty} d(w - kw_s)$$







# Impulse-Train Sampling

- The convolution of a signal with an impulse shifts the signal, i.e.,

$$X(jw) * d(w - w_0) = X(j(w - w_0))$$

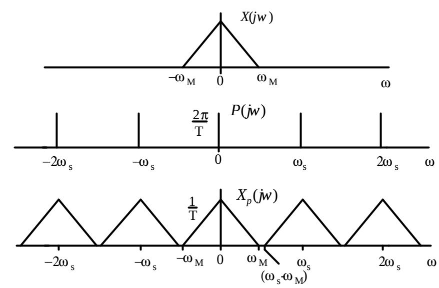
Thus, the convolution of a signal with an impulse copies the signal to the location of the impulse

- Now, in the convolution $X(jw)*P(jw)$, $P(jw)$ is a periodic impulse train;
- Then, it follows that the Fourier transform of the sampled signal is periodic, i.e.,

$$X_p(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(w - kw_s))$$



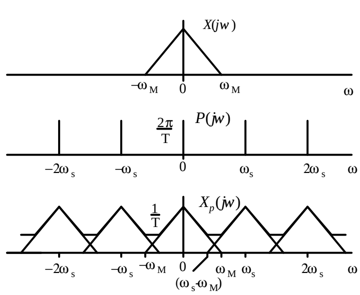

# Spectrum of Sampled Signal with $w_s > 2w_M$







# Spectrum of Sampled Signal with $w_s < 2w_M$







# Sampling Theorem

- Let $x(t)$ be a bandlimited signal with $X(jw) = 0$ for $|w| > w_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

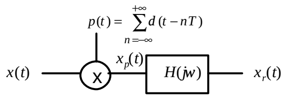
$$w_s > 2w_M, \quad \text{where} \quad w_s = \frac{2p}{T}$$

- Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values.

  This impulse train is then processed through an ideal lowpass filter with gain $T$ and cutoff frequency greater than $w_M$ and less than $w - w_M$.

  The resulting output signal will be exactly equal to $x(t)$

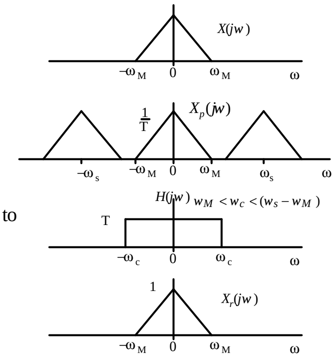
# Sampling Process




- Sampling process is modeled by multiplying the continuous-time signal $x(t)$ with a periodic impulse train $p(t)$
- The recovered signal $x_r(t)$ is obtained by lowpass filtering the sampled signal $x_p(t)$



- Spectrum for $x(t)$
- Corresponding spectrum for $x_p(t)$
- Ideal lowpass filter to recover $X(jw)$ from $X_p(jw)$
- Spectrum of $x_r(t)$





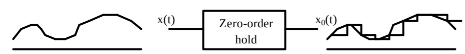

# Sampling Theorem

- If the sampling frequency at least twice as high as the highest frequency component of the bandlimited signal, i.e., $w_s > 2w_M$ , then the original signal can be recovered from its samples
- If the above condition is not fulfilled, i.e., the frequency components above $w_s/2$ will be ***aliased*** into the band of interest $|w| < w_M$
- The frequency $2w_M$ which must be exceeded by the sampling frequency is commonly referred to as the ***Nyquist frequency*** or ***Nyquist rate*** (and $w_M$ as ***one-half the Nyquist rate***)
- The frequency $w_s/2$ is referred to as the ***folding frequency***
- ***Critical sampling*** corresponds to $w_s = 2w_M$
- ***Undersampling*** corresponds to $w_s < 2w_M$
- ***Oversampling*** corresponds to $w_s >> 2w_M$



# Sampling with a Zero-Order Hold

- In impulse-train sampling, narrow and large-amplitude pulses which approximate impulses are in practice difficult to generate and transmit
- Thus, it is often more convenient to generate the sampled signal in a form referred to as a ***zero-order hold***
- Such a system samples $x(t)$ at a given instant and holds that value until the next instant

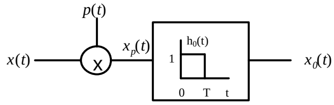



- The reconstruction of $x(t)$ from the output of a zero-order hold can again be carried out by lowpass filtering
- However, the lowpass filter has no more a constant gain in the passband



# Sampling with a Zero-Order Hold

- The output $x_0(t)$ of the zero-order hold can be be generated by impulse-train sampling followed by an LTI system with rectangular impulse response

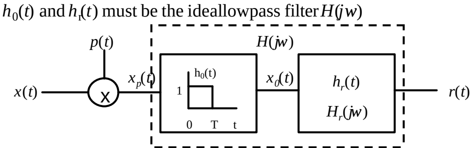



- The frequency response of $h_0(t)$ is the ***sinc*** function, i.e., it has a lowpass characteristic in frequency domain
- In order to reconstruct $x(t)$ from $x_0(t)$ perfectly, the response of the reconstruction filter $h_r(t)$ must compensate for the performance of $h_0(t)$



# Sampling with a Zero-Order Hold

- For perfect reconstruction, i.e., $r(t) = x(t)$, the cascade combination of $h_0(t)$ and $h_r(t)$ must be the ideal lowpass filter $H(jw)$



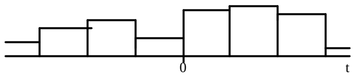

- The cascade combination is: $H(jw) = H_0(jw)H_r(jw)$,

  where $H_0(jw) = e^{-jwT/2}\left[\frac{2\sin(wT/2)}{w}\right]$

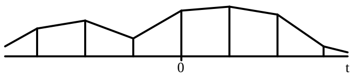
- The reconstruction filter is: $H_r(jw) = \frac{e^{jwT/2}H(jw)}{2\sin(wT/2)/w}$

## Reconstruction of a Signal from Its Samples

- ***Interpolation*** can be used to reconstruct the signal from its samples
- ***Zero-order hold*** is a simple interpolation procedure:




- The zero-order-hold interpolation is a very rough approximation of the desired transfer function of the exact interpolating filter
- ***Linear interpolation*** is another useful interpolation procedure:






## Reconstruction of a Signal from Its Samples

- The interpretation of the reconstruction of $x(t)$ as a process of interpolation can be investigated in the time domain

$$x_r(t) = x_p(t) * h(t)$$

or

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT)$$

- For the ideallowpass filter $H(jw)$, the impulse response is:

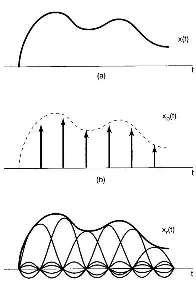
$$h(t) = \frac{w_c T \sin(w_c T)}{p w_c T}$$

so that

$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{p} \frac{\sin(w_c(t-nT))}{w_c(t-nT)}$$



## Reconstruction of a Signal from Its Samples




$$x_r(t) = \sum_{n=-\infty}^{+\infty} x(nT) \frac{w_c T}{p} \frac{\sin(w_c(t-nT))}{w_c(t-nT)}$$

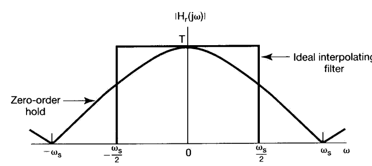
- The reconstruction according to the above equation with $w_c = w_s/2$, i.e., $x_r(t)$ is formed as a superposition of shifted ***sinc*** functions weighted by the values of $x(nT)$

=> ***band-limited interpolation***



## Reconstruction of a Signal from Its Samples

- Transfer function for the zero-order-hold and for the ideal interpolating filter:

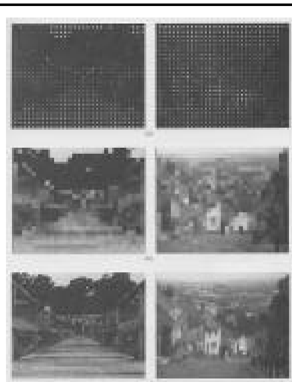



- The zero-order-hold interpolation is a very rough approximation of the desired transfer function of the exact interpolating filter



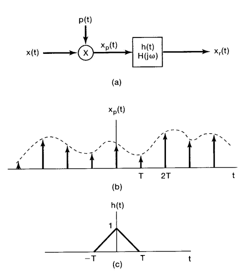
## Example: Image sampling

a) Impulse sampling of the images
b) Zero-order hold applied to pictures in a)
c) Impulse sampling and zero-order hold with one-third the horizontal and vertical spacing used in a) and b), i.e., the sample spacing is reduced



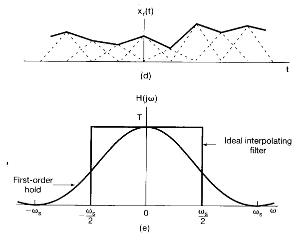


## Linear Interpolation (First-Order Hold)




- Linear interpolation (first-order hold) represented as impulse-train sampling followed by convolution with a triangular impulse response
  - (a) System for sampling and reconstruction
  - (b) Impulse train of samples
  - (c) Impulse response of a first-order hold

## Linear Interpolation (First-Order Hold)




(d) First-order hold applied to the sampled signal

(e) Comparison of the transfer function of ideal interpolationg filter and first-order hold

$$H(jw) = \frac{1}{T}\left[\frac{\sin(wT/2)}{w/2}\right]^2$$



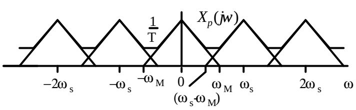
## Linear Interpolation (First-Order Hold)

- Result of applying a first-order hold after impulse sampling with one-third the horizontal and vertical spacing used in Fig. 7.12 (a) and (b)
- The resulting image is smoothed at the edges



## The Effect of Undersampling: Aliasing

- If the requirement of the sampling theorem is not met, i.e., $w_s < 2w_M$, the spectrum, $X(jw)$, of the original continuous-time signal, $x(t)$, can no more be recovered from the periodic spectrum $X_p(jw)$ using a lowpass filter




***ALIASING***

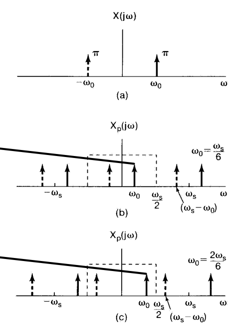


## Example: Signal Frequency $w_0 < w_s/2$ => No Aliasing

(a) Spectrumof the original CT signal

(b) $w_0 = \frac{w_s}{6}$; $x_r(t) = \cos w_0 t = x(t)$

(c) $w_0 = \frac{2w_s}{6}$; $x_r(t) = \cos w_0 t = x(t)$

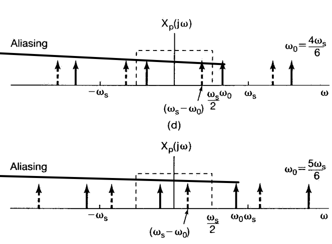





## Example: Signal Frequency $w_0 > w_s/2$ => Aliasing

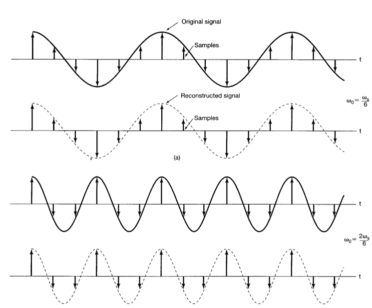
(d) $w_0 = \frac{4w_s}{6}$; $x_r(t) = \cos w_0 t \neq x(t)$

(e) $w_0 = \frac{5w_s}{6}$; $x_r(t) = \cos w_0 t \neq x(t)$





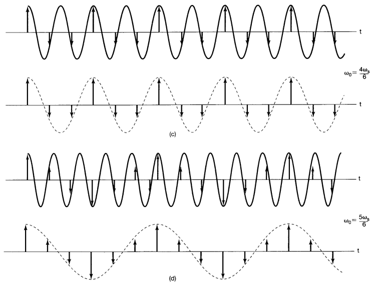

## Effect of Aliasing on a Sinusoidal Signal $w_0 < w_s/2$




***No aliasing***

***No aliasing***

## Effect of Aliasing on a Sinusoidal Signal $w_0 > w_s/2$




$$\frac{4w_s}{6} = \frac{w_s}{2} + \frac{w_s}{6} => \frac{2w_s}{6}$$

=> ***Aliasing***

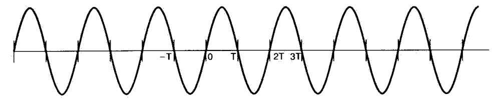
$$\frac{5w_s}{6} = \frac{w_s}{2} + \frac{2w_s}{6} => \frac{w_s}{6}$$

=> ***Aliasing***



## Example : Critical Samplig

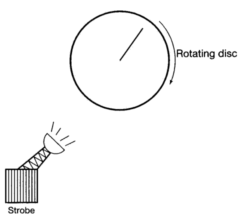
- Critical samplingof a sinusoidal signal $w_{signal} = ½ w_s$




- As can be seen, the signal cannot be recovered from the samples



## Example: Undersampling




- The effect of undersampling is illustrated by the strobe effect:
  - $w_{strobe} >> w_{disc}$ => the speed of rotation is perceived correctly
  - $w_{strobe} < ½ w_{disc}$ => the rotation appears to be at lower frequency
- Furthermore, because of phase reversal, the direction of rotation appears to be wrong
  - $w_{strobe} = w_{disc}$ the radial line on the disc appears stationary , i.e., the rotational frequency of the disc and its harmonics have been aliased to zero frequency



## Summary

- The sampling theorem explicitly requires that the sampling rate is greater than twice the highest frequency in the signal
- In practice, an antialiasing filter is required before sampling in order to guarantee the elimination of high frequency components from the signal
- In digital processing of signal samples, the computations required for generation of one output sample must be completed within the sampling period $T$
- The sampling frequency determines the computational requirements of the DSP implementation
- Thus, oversampling, i.e., increasing the sampling rate considerably above the required minimum, results in higher computational requirements