The Discrete-Time Fourier Transform

- The Fourier series representation of a discrete-time periodic signal is a finite series, as opposed to the infinite series representation required for the continuous-time periodic signals.
- The discrete-time Fourier analysis is discussed.
- The differences between continuous-time and discrete-time Fourier transforms are considered (similar to those between CT and DT Fourier series).

Development of the Discrete-Time Fourier Transform

- An aperiodic signal $x(t)$ was earlier (Chapter 4) represented by first constructing a periodic signal $x_p(t)$ that was equal to $x(t)$ over one period.
- The Fourier series representation for $x_p(t)$ converged to the Fourier transform representation for $x(t)$.
- The similar procedure is applied to discrete-time signals in order to develop the Fourier transform representation for discrete-time aperiodic sequences.

Representation of Aperiodic Signals

- Let us examine the effect on the Fourier series representation of $x[n]$.
- **Fourier series**:

  $$x_p[n] = \sum_{k} a_k N \delta[k - N_n]$$

  $$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2 \pi k n / N}$$

- Since $x[n]=x_p[n]$ over a period that includes the interval $-N/2 \leq n \leq N/2$, it is convenient to choose the interval $0 \leq n \leq N-1$ in the summation, so that $x[n]$ can be replaced by $x[n]$ in the summation.

  $$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j 2 \pi k n / N}$$

- We can now express $x_p[n]$ in terms of $X(e^{j \omega})$ as

  $$x_p[n] = \sum_{k} a_k N \delta[n - k N]$$

- Defining the function

  $$X(e^{j \omega}) = \sum_{n} x[n] e^{-j 2 \pi \omega n / N}$$

  we see that the coefficients $a_k$ are proportional to samples of $X(e^{j \omega})$.

  $$a_k = \frac{1}{N} X(e^{j \omega})$$

  where $\omega_k = 2 \pi / N$.

  We can now express $x_p[n]$ in terms of $X(e^{j \omega})$ as

  $$x_p[n] = \sum_{k} \frac{1}{N} X(e^{j \omega_k}) \delta[n - k N]$$
Representation of Aperiodic Signals

- Equivalently, since \( 2\pi N = \omega_0 \)

\[ s_p[n] = \frac{1}{2\pi} \sum_{k=-N}^{N} X(e^{j\omega_0 k}) e^{j\omega_0 k} \]

- As \( N \) increases, \( \omega_0 \) decreases, and as \( N \) approaches infinity, and the summation passes to an integral

- As \( N \) approaches infinity, \( s_p[n] \rightarrow x[n] \) and the above equation becomes

\[ s_p[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega_0}) e^{j\omega_0 n} d\omega \]

The Discrete-Time Fourier Transform

- The discrete-time Fourier transform is periodic and there is finite interval of integration in the synthesis equation

- Discrete-time complex exponentials that differ in frequency by a multiple of \( 2\pi \) are identical

- Periodicity of \( e^{j\omega_0} \):
  \( \omega = 0 \) and \( \omega = 2\pi \) yield the same signal

Example 5.1: Exponential Sequences

- Consider the signal \( x[n] = a^n u[n], \ |a|<1 \)

- The Fourier transform is given by

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}} \]

- For \( a > 0 \), the system corresponds to a lowpass filter

- For \( a < 0 \), the system corresponds to a highpass filter

Example 5.1: Lowpass Filter, \( a > 0 \)

- Signals at frequencies near \( \omega = 0 \) and any even multiple of \( 2\pi \) are slowly varying

- High frequencies in discrete-time are the values of \( \omega \) near odd multiples of \( \pi \)
Example 5.1: Highpass Filter, \( \alpha < 0 \)

\[
\begin{align*}
|H(e^{j\omega})| &= \frac{\alpha}{\alpha^2 + (\omega - \omega_c)^2} \\
\omega_c &= \text{cutoff frequency}
\end{align*}
\]

Example 5.2:

- Consider the signal \( |x[n]| = \alpha^n \), \( |x| < 1 \)

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} |x[n]| e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \alpha^n e^{-j\omega n}
\]

Example 5.3: Rectangular Pulse

- Consider the rectangular pulse

\[
x[n] = \begin{cases} 
1, & |n| \leq N_1 \\
0, & |n| > N_1 
\end{cases}
\]

\[
X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{\sin(N_1 + \frac{1}{2})}{\sin(\frac{1}{2})}
\]

- The function is the discrete-time counterpart of the sinc function
- This function, however, is periodic with period \( 2\pi \), whereas the sinc function is aperiodic

Example 5.4: Unit Impulse

- Let \( \delta[n] \) be a unit impulse \( x[n] = \delta[n] \)
- The analysis equation gives \( X(e^{j\omega}) = 1 \)
- The unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies
- We obtain now

\[
x[n] = \frac{1}{2\pi} \int_0^{2\pi} e^{j\omega n} d\omega = \frac{\sin(W)}{\pi n}
\]

- The frequency of the oscillations in the approximation increases as \( W \) is increased
- The amplitude of these oscillations decreases relative to the magnitude of \( x[0] \) as \( W \) is increased, and the oscillations disappear for \( W = \pi \)
Example 5.4: Approximation of the Unit Sample

The Fourier Transform for Periodic Signals

- Consider the signal $x[n] = e^{j\omega_0 n}$.
- In continuous-time, the Fourier transform of $e^{j\omega_0 t}$ was interpreted as an impulse at $\omega = \omega_0$.
- The discrete-time Fourier transform must be periodic in $\omega$ with period $2\pi$.
- Thus, the Fourier transform of $x[n]$ should have impulses at $\omega = \omega_0$, $\omega = \omega_0 + 2\pi$, $\omega = \omega_0 + 4\pi$, etc., i.e., $X(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k)$.

Example 5.5: Periodic (Sinusoidal) Signal

- Consider the periodic signal $x[n] = a_0 + a_N e^{j2\pi/N} \cos \left( 2\pi \frac{n}{N} \right) + \cdots + a_N e^{j2\pi/N} \sin \left( 2\pi \frac{n}{N} \right)$.
- $x[n]$ is a linear combination of signals with $\omega_0 = 0$, $\omega = 2\pi/N$, $\omega = 4\pi/N$, ..., $\omega = (N-1)2\pi/N$.

The Fourier Transform for Periodic Signals

- Now, consider a periodic sequence $x[n]$ with period $N$ and with the Fourier series
  $x[n] = \sum_{k=0}^{N-1} a_k e^{j2\pi k/N}$.  

- The Fourier transform is $X(\omega) = \sum_{k=0}^{N-1} a_k \delta(\omega - 2\pi k/N)$.

- The Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients.
- This can be verified by noting that $x[n]$ is the linear combination of complex exponentials and so is the Fourier transform.

Example 5.5: Periodic (Sinusoidal) Signal

- Consider the periodic signal $x[n] = \cos(n\omega_0) + \frac{1}{2} \cos(2n\omega_0) + \cos(3n\omega_0)$, with $\omega_0 = \frac{\pi}{5}$.

- The Fourier transform is
  $X(\omega) = \sum_{k=0}^{N-1} a_k \delta(\omega - k\omega_0)$.

- This can be verified by noting that $x[n]$ is the linear combination of complex exponentials and so is the Fourier transform.
Consider the discrete-time counterpart of the periodic impulse train

\[ x[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN) \]

Fourier series coefficients:

\[ \alpha_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \]

\[ X(e^{jw}) = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} \]

\[ = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} \]

Example 5.6: Periodic Impulse Train

Consider the discrete-time counterpart of the periodic impulse train

\[ x[n] = \sum_{k=-\infty}^{\infty} \delta(n - kN) \]

Fourier series coefficients:

\[ \alpha_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} = \frac{1}{N} \]

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\[ = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} kn} \]

Properties of the Discrete-Time Fourier Transform

- Periodicity: \( X(e^{j(\theta + \omega)}) = X(e^{j\theta}) \)

- Linearity: \( aX_1[n] + bX_2[n] \rightarrow aX_1(e^{j\theta}) + bX_2(e^{j\theta}) \)

- Time Shifting: \( X[n - n_0] \rightarrow e^{-j\omega n_0} X(e^{j\theta}) \)

- Frequency Shifting: \( e^{j\omega_0 n}x[n] \rightarrow X(e^{j(\theta - \omega_0)}) \)

Time Expansion

- Let \( k \) be a positive integer, and define the signal

\[ x_k[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases} \]

- For \( k=3 \), \( x_3[n] \) is obtained from \( x[n] \) by placing \( k-1 \) zeros between successive values of the original signal

- Intuitively, we can think of \( x_k[n] \) as a slow-down version of \( x[n] \)

Time Expansion

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Convolution Property

\[ x[n] \rightarrow h[n] \rightarrow y[n] = x[n] * h[n] \]

In frequency domain:

\[ Y(e^{j\theta}) = X(e^{j\theta})H(e^{j\theta}) \]

Convolution in the time domain corresponds to multiplication in the frequency domain

- The frequency response \( H(e^{j\omega}) \) captures the change in complex amplitude of the Fourier transform of the input at frequency \( \omega \)
Example 5.11: Delay on \( n_0 \) Samples

- Impulse response: \( h[n] = \delta[n-n_0] \)
- Frequency response: \( H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} = e^{-j\omega n_0} \)
- Thus, for any input \( x[n] \) with the Fourier transform \( X(e^{j\omega}) \), the Fourier transform of the output is
  \[ Y(e^{j\omega}) = e^{-j\omega n_0}X(e^{j\omega}) \]

Example 5.12: Ideal Lowpass Filter

- Impulse response: \( h[n] = \frac{1}{2\pi/\omega_0} \sin\left(\frac{\pi}{\omega_0}n\right) \)
- The impulse response is a sinc function
- The impulse response is not causal and its oscillatory behavior is not desired

Example 5.14: Ideal Bandstop Filter Based on Ideal Lowpass Filters and the Frequency Shifting Property

- Frequency shifting:
  \[ W(e^{j\omega}) = H_p(e^{j\omega}) X(e^{j\omega}) \]
- Convolution:
  \[ W(e^{j\omega}) = H_p(e^{j\omega}) X(e^{j\omega}) \]
- Frequency shifting:
  \[ W(e^{j\omega}) = H_p(e^{j\omega}) X(e^{j\omega}) \]
- Periodicity:
  \[ W(e^{j\omega}) = H_p(e^{j\omega}) X(e^{j\omega}) \]

Example 5.14: continued...

- Convolution:
  \[ W_1(e^{j\omega}) = H_1(e^{j\omega}) X(e^{j\omega}) \]
- Linearity:
  \[ Y(e^{j\omega}) = [H_1(e^{j\omega}) + H_2(e^{j\omega})] X(e^{j\omega}) \]
- Frequency response:
  \[ H(e^{j\omega}) = [H_1(e^{j\omega}) + H_2(e^{j\omega})] \]
- Bandstop filter

The Multiplication Property

- Consider a product of two sequences
  \[ y[n] = x_1[n] x_2[n] \]
- The corresponding Fourier transforms:
  \[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n] x_2[-n] e^{-j\omega n} \]
  \[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n] x_2[-n+2\pi] e^{-j\omega n} \]
  \[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} X_1(e^{j\omega}) X_2(e^{j\omega}) e^{-j\omega n} \]
The Multiplication Property

- The bracketed summation is the discrete-time Fourier transform

\[ X_2(e^{jw}) = \left[ \sum_{n=-\infty}^{\infty} x[n] e^{-jwn} \right] \]

and we have

\[ Y(e^{jw}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{jw}) X_2(e^{jw}) \, dw \]

This corresponds to a periodic convolution of \( X_1(e^{jw}) \) and \( X_2(e^{jw}) \) and the integral is evaluated over any interval of length \( 2\pi \).

Summary of Fourier Series and Transform Properties

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Systems Characterized by Linear Constant-Coefficient Difference Equations

- An LTI system with input \( x[n] \) and output \( y[n] \) is described by

\[ \sum_{k=0}^{N} a_k x[n-k] + \sum_{k=0}^{M} b_k y[n-k] = 0 \]

- The frequency response of the system, \( H(e^{jw}) \), can be determined by applying the Fourier transform to both sides of the difference equation and using the linearity and time-shifting operations

\[ \sum_{k=0}^{N} a_k e^{-jkw} X(e^{jw}) + \sum_{k=0}^{M} b_k e^{-jkw} Y(e^{jw}) = 0 \]

where \( X(e^{jw}) \to \overset{F}{\longrightarrow} x[n] \) and \( Y(e^{jw}) \to \overset{F}{\longrightarrow} y[n] \)

Example 5.18: First Order IIR Filter

- Consider the first order recursive or infinite impulse response (IIR) filter

\[ y[n] - ay[n-1] = x[n], \quad |a| < 1 \]

- The frequency response of this system is

\[ H(e^{jw}) = \frac{1}{1 - ae^{-jw}} \]

- The impulse response is calculated earlier: \( h[n] = a^n u[n] \)

Example 5.19: Second Order IIR Filter

- The difference equation is

\[ y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n] \]

- The frequency response is

\[ H(e^{jw}) = \frac{2}{1 - \frac{3}{4} e^{-jw} + \frac{1}{8} e^{-j2w}} \]

- Factoring the denominator cascade of two 1^st order sections

\[ H(e^{jw}) = \left[ 1 - 0.5 e^{-jw} \right] \frac{1}{1 - \frac{1}{2} e^{-jw}} \]

- Partial fraction expansion parallel interconnection of two 1^st order sections

\[ H(e^{jw}) = \frac{4}{1 - e^{-jw}} \frac{1}{1 - \frac{1}{2} e^{-jw}} \]

- The impulse response is

\[ h[n] = \frac{3}{2} \delta[n] - \frac{3}{4} \delta[n-1] + \frac{1}{2} \delta[n-2] \]