The Discrete-Time Fourier Transform

The Discrete-Time Fourier Transform

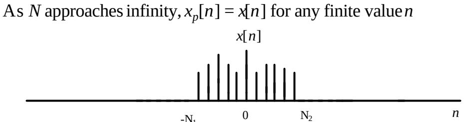
- **The Fourier series representation of a discrete-time periodic signal is a finite series, as opposed to the infinite series representation required for the continuous-time periodic signals**
- The discrete-time Fourier analysis is discussed
- The differences between continuous-time and discrete-time Fourier transforms are considered (similar to those between CT and DT Fourier series)

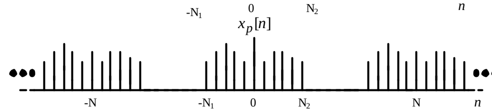
Development of the Discrete-Time Fourier Transform

- An aperiodic signal $x(t)$ was earlier (Chapter 4) represented by first constructing a periodic signal $x_p(t)$ that was equal to $x(t)$ over one period
- The Fourier series representation for $x_p(t)$ converged to the Fourier transform representation for $x(t)$
- The similar procedure is applied to discrete-time signals in order to develop the Fourier transform representation for discrete-time aperiodic sequences

Representation of Aperiodic Signals

- Consider a general sequence $x[n]$ that is of ***finite duration***, i.e., for some integers N_1 and N_2, $x[n] = 0$ outside the range $-N_1 \leq n \leq N_2$
- A periodic signal $x_p[n]$ is constructed for which $x[n]$ is one period
- As N approaches infinity, $x_p[n] = x[n]$ for any finite value n





Representation of Aperiodic Signals

- Let us examine the effect on the Fourier series representation of $x_p[n]$
- ***Fourier series:***

$$x_p[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2p/N)n}$$

$$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x_p[n] e^{-jk(2p/N)n}$$

- Since $x[n]=x_p[n]$ over a period that includes the interval $-N_1 \leq n \leq N_2$, it is convenient to choose the interval of summation to include this interval, so that $x_p[n]$ can be replaced by $x[n]$ in the summation

$$a_k = \frac{1}{N}\sum_{k=-N_1}^{N_2} x[n] e^{-jk(2p/N)n} = \frac{1}{N}\sum_{k=-\infty}^{+\infty} x[n] e^{-jk(2p/N)n}$$

Representation of Aperiodic Signals

- Defining the function

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-jwn}$$

we see that the coefficients a_k are proportional to samples of $X(e^{jw})$

$$a_k = \frac{1}{N} X(e^{jkw_0}) \qquad \text{where } w_0 = 2p/N$$

- We can now express $x_p[n]$ in terms of $X(e^{jw})$ as

$$x_p[n] = \sum_{k=\langle N\rangle} \frac{1}{N} X(e^{jkw_0}) e^{jkw_0 n}$$

Representation of Aperiodic Signals

- Equivalently, since $2p/N=w_0$

$$x_p[n]=\frac{1}{2p}\sum_{k=\langle N\rangle} X(e^{jkw_0})e^{jkw_0 n}w_0$$

- As N increases w_0 decreases, and as N approaches infinity, and the summation passes to an integral
- As N approaches infinity, $x_p[n]$ -> $x[n]$ and the above equation becomes

$$x_p[n]=\frac{1}{2p}\int_{2p} X(e^{jw})e^{jwn}dw$$

The Discrete-Time Fourier Transform

$$\boxed{x[n]=\frac{1}{2p}\int_{2p} X(e^{jw})e^{jwn}dw}$$

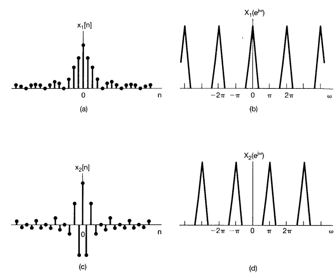
$$\boxed{X(e^{jw})=\sum_{n=-\infty}^{+\infty} x[n]e^{-jwn}}$$

The Discrete-Time Fourier Transform

- **The discrete-time Fourier transform is periodic and there is finite interval of integration in the synthesis equation**
- **Discrete-time complex exponentials that differ in frequency by a multiple of 2*p* are identical**
- Periodicity of e^{jwn} :
  $w = 0$ and $w = 2p$ yield the same signal

The Discrete-Time Fourier Transform

- Signals at frequencies near w=0 and any even multiple of of $2p$ are slowly varying
- High frequencies in discrete-time are the values of w near odd multiples of p



Example 5.1: Exponential Sequences

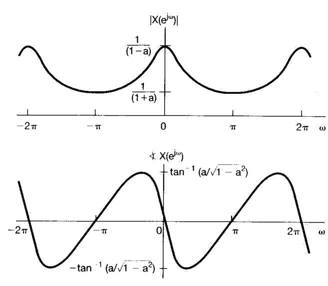
- Consider the signal $x[n]=a^n u[n], \quad |a|<1$
- The Fourier transform is given by

$$X(e^{jw})=\sum_{n=-\infty}^{+\infty} a^n u[n]\, e^{-jwn}$$

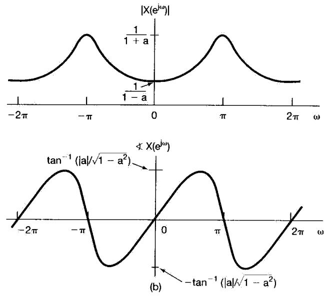
$$=\sum_{n=0}^{+\infty}\left(ae^{-jw}\right)^n=\frac{1}{1-ae^{-jw}}$$

- For $a > 0$, the system corresponds to a lowpass filter
- For $a < 0$, the system corresponds to a highpass filter

Example 5.1: Lowpass Filter, $a > 0$



Example 5.1: Highpass Filter, $a < 0$



Example 5.2:

- Consider the signal $x[n] = a^{|n|}, \quad |a| < 1$

$$X(e^{jw}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-jwn} = \sum_{n=0}^{+\infty} a^{n} e^{-jwn} + \sum_{n=-\infty}^{-1} a^{-n} e^{-jwn}$$

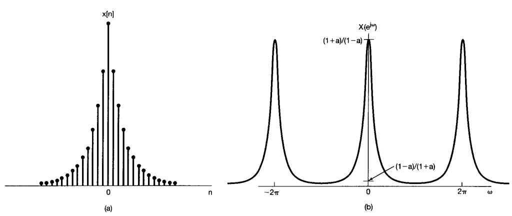
$$X(e^{jw}) = \sum_{n=0}^{+\infty} \left(ae^{-jw}\right)^{n} + \sum_{m=1}^{+\infty} \left(ae^{jw}\right)^{m}$$

$$X(e^{jw}) = \frac{1}{1-ae^{-jw}} + \frac{ae^{jw}}{1-ae^{jw}} = \frac{1-a^2}{1-2a\cos w + a^2}$$

- In this case, $X(e^{jw})$ is real !

Example 5.2:

Signal $x[n]=a^{|n|}$ and its Fourier transform



$0 < a < 1$

Example 5.3: Rectangular Pulse

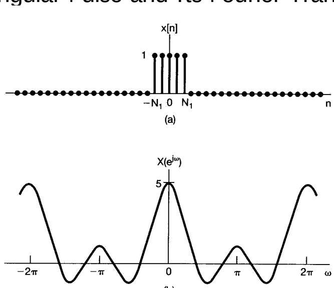
- Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$

$$X(e^{jw}) = \sum_{n=-N_1}^{+N_1} e^{-jwn} = \frac{\sin w(N_1 + ½)}{\sin(w/2)}$$

- The function is the discrete-time counterpart of the sinc function
- This function, however, is periodic with period $2p$, whereas the sinc function is aperiodic

Example 5.3:
Rectangular Pulse and Its Fourier Transform



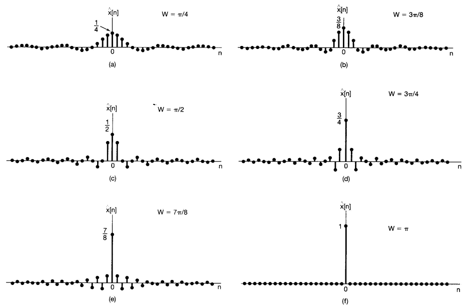
Example 5.4: Unit Impulse

- Let $x[n]$ be a unit impulse $x[n] = d[n]$
- The analysis equation gives $X(e^{jw}) = 1$
- The unit impulse has a Fourier transform representation consisting of equal contributions at all frequencies
- We obtain now

$$\hat{x}[n] = \frac{1}{2p}\int_{-W}^{W} e^{jwn}\,dw = \frac{\sin Wn}{pn}$$

- The frequency of the oscillations in the approximation increases as W is increased
- The amplitude of these oscillations decreases relative to the magnitude of $x[0]$ as W is increased, and the oscillations disappear for $W = p$

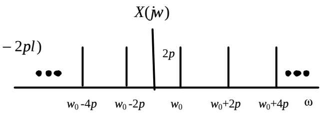
Example 5.4: Approximation of the Unit Sample



The Fourier Transform for Periodic Signals

- Consider the signal $x[n] = e^{jw_0 n}$
- In continuos-time, the Fourier transform of $e^{jw_0 t}$ was interpreted as an impulse at $w = w_0$
- The discrete-time Fourier transform must be periodic in w with period $2p$
- Thus, the Fourier transform of $x[n]$ should have impulses at w_0, w_0+2p, w_0+4p, etc., i.e.,

$$X(e^{jw}) = \sum_{l=-\infty}^{+\infty} 2p\ d(w - w_0 - 2pl)$$



The Fourier Transform for Periodic Signals

- Substituting into the synthesis equation

$$x[n] = \frac{1}{2p}\int_{2p} X(e^{jw})e^{jwn}dw = \frac{1}{2p}\int_{2p}\sum_{l=-\infty}^{+\infty} 2p\ d(w - w_0 - 2pl)\ e^{jwn}dw$$

- Any interval of length $2p$ includes exactly one impulse, then if the interval of integration chosen includes the impulse at w_0+2pr, we have

$$x[n] = \frac{1}{2p}\int_{2p} X(e^{jw})e^{jwn}dw = e^{j(w_0+2pr)n} = e^{jw_0 n}$$

The Fourier Transform for Periodic Signals

- Now, consider a periodic sequence $x[n]$ with period N and with the Fourier series

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2p/N)n}$$

- The Fourier transform is

$$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} 2p\ a_k d(w - \frac{2pk}{N})$$

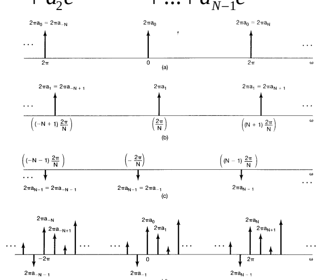
- The Fourier transform of a periodic signal can be directly constructed from its Fourier coefficients
- This can be verified by noting that $x[n]$ is the linear combination of complex exponentials and so is the Fourier transform

The Fourier Transform for Periodic Signals

$$x[n] = a_0 + a_1 e^{j(2p/N)n} + a_2 e^{j2(2p/N)n} + \ldots + a_{N-1}e^{j(N-1)(2p/N)n}$$

- $x[n]$ is a linear combination of signals with

$$w_0 = 0,\quad \frac{2p}{N},\quad \frac{4p}{N},\ldots\quad \frac{(N-1)2p}{N}$$



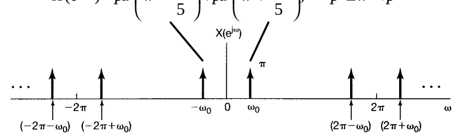
Example 5.5: Periodic (Sinusoidal) Signal

- Consider the periodic signal

$$x[n] = \cos w_0 n = \tfrac{1}{2}e^{jw_0 n} + \tfrac{1}{2}e^{-jw_0 n}, \quad \text{with } w_0 = \frac{2p}{5}$$

$$X(e^{jw}) = \sum_{l=-\infty}^{+\infty} pd\left(w - \frac{2p}{5} - 2pl\right) + \sum_{l=-\infty}^{+\infty} pd\left(w + \frac{2p}{5} - 2pl\right)$$

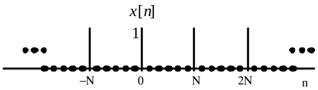
$$X(e^{jw}) = pd\left(w - \frac{2p}{5}\right) + pd\left(w + \frac{2p}{5}\right), \quad -p \le w < p$$



Example 5.6: Periodic Impulse Train

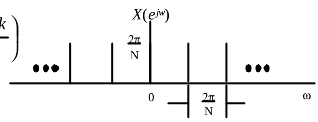
Consider the discrete-time counterpart of the periodic impulse train

$$x[n] = \sum_{k=-\infty}^{+\infty} \delta(n - kN)$$



Fourier series coefficients: $a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk(2\pi/N)n} = \frac{1}{N}$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



Properties of the Discrete-Time Fourier Transform

- <u>Periodicity:</u> $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Fourier transform pairs: $x_1[n] \xleftrightarrow{F} X_1(e^{j\omega})$, $x_2[n] \xleftrightarrow{F} X_2(e^{j\omega})$

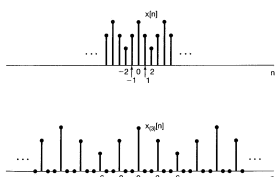
- <u>Linearity:</u> $ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
- <u>Time Shifting:</u> $x[n-n_0] \xleftrightarrow{F} e^{-j\omega n_0} X(e^{j\omega})$
- <u>Frequency Shifting:</u> $e^{j\omega_0 n} x[n] \xleftrightarrow{F} X(e^{j(\omega-\omega_0)})$

Time Expansion

- Let k be a positive integer, and define the signal

$$x_k[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

- For k=3, $x_{(k)}[n]$ is obtained from $x[n]$ by placing k-1 zeros between successive values of the original signal
- Intuitively, we can think of $x_{(k)}[n]$ as a slowed-down version of $x[n]$



Time Expansion

- Since $x_{(k)}[n]$ equals 0 unless n is a multiple of k, i.e., unless n=rk, the Fourier transform of $x_{(k)}[n]$ can be given as

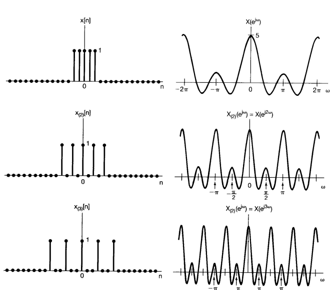
$$X_k(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_{(k)}[rk] e^{-j\omega rk}$$

- Furthermore, since $x_{(k)}[rk]$=$x[r]$, the Fourier transform can be written as

$$X_k(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[r] e^{-j(k\omega)r} = X(e^{jk\omega})$$

$$\boxed{x_{(k)}[n] \xleftrightarrow{F} X(e^{jk\omega})}$$

Time Expansion



- Note that bas the signal is spread out and slowed down in time by taking k>1 its Fourier transform is compressed
- The application of the time scaling is in increasing and decreasing the sampling rate

Convolution Property

$x[n]$ — $\boxed{h[n]}$ — $y[n]$= $x[n]$*$h[n]$

In frequency domain:

$$\boxed{Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega})}$$

Convolution in the time domain corresponds to multiplication in the frequency domain

- The frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at frequency ω

Example 5.11: Delay on n_0 Samples

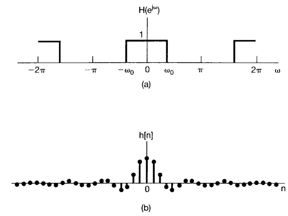
- Impulse response: $h[n] = d[n - n_0]$
- Frequency response: $H(e^{jw}) = \sum_{n=-\infty}^{+\infty} d[n - n_0]\; e^{-jwn} = e^{-jwn_0}$
- Thus, for any input $x[n]$ with the Fourier transform $X(e^{jw})$, the Fourier transform of the output is

$$Y(e^{jw}) = e^{-jwn_0} X(e^{jw})$$

$$y[n] = x[n - n_0]$$

Example 5.12: Ideal Lowpass Filter

$$h[n] = \frac{1}{2p} \int_{-p}^{+p} H(e^{jw}) e^{jwn}\, dw = \frac{1}{2p} \int_{-w_c}^{+w_c} e^{jwn} dw = \frac{\sin w_c n}{pn}$$

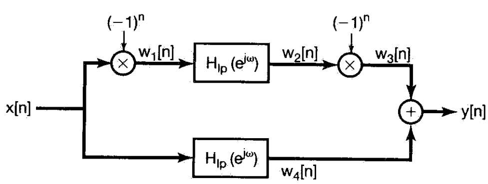


- The impulse response is a sinc function

$$h[n] \neq 0, \quad \text{for } n < 0$$

- The impulse response is not causal and its oscillatory behavior is not desired

Example 5.14:
Ideal Bandstop Filter Based on Ideal Lowpass Filters and the Frequency Shifting Property



Example 5.14: Ideal bandstop filter based on ideal lowpass filters and the frequency shifting property

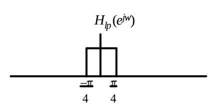
Frequency shifting: $w_1[n] = (-1)^n x[n] = e^{jpn} x[n]$

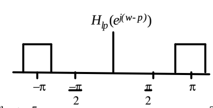
$W_1(e^{jw}) = X(e^{j(w-p)})$

Convolution: $W_2(e^{jw}) = H_{lp}(e^{jw}) X(e^{j(w-p)})$

Frequency shifting: $W_3(e^{jw}) = W_2(e^{j(w-p)}) = H_{lp}(e^{j(w-p)}) X(e^{j(w-2p)})$

Periodicity: $W_3(e^{jw}) = H_{lp}(e^{j(w-p)}) X(e^{jw})$





Example 5.14: continued...

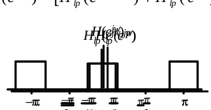
Convolution: $W_4(e^{jw}) = H_{lp}(e^{jw}) X(e^{jw})$

Linearity: $Y(e^{jw}) = W_3(e^{jw}) + W_4(e^{jw})$

$$= \left[H_{lp}(e^{j(w-p)}) + H_{lp}(e^{jw})\right] X(e^{jw})$$

Frequency response: $H(e^{jw}) = \left[H_{lp}(e^{j(w-p)}) + H_{lp}(e^{jw})\right]$

Bandstop filter



The Multiplication Property

- Consider a product of two sequences $y[n] = x_1[n] x_2[n]$
- The corresponding Fourier transforms: $Y(e^{jw}),\ X_1(e^{jw}),\ X_2(e^{jw})$

$$Y(e^{jw}) = \sum_{n=-\infty}^{+\infty} y[n] e^{-jwn} = \sum_{n=-\infty}^{+\infty} x_1[n] x_2[n] e^{-jwn}$$

$$Y(e^{jw}) = \sum_{n=-\infty}^{+\infty} \left\{ \frac{1}{2p} \int_{2p} X_1(e^{jq}) e^{jqn} dq \right\} x_2[n] e^{-jwn}$$

$$Y(e^{jw}) = \frac{1}{2p} \int_{2p} X_1(e^{jq}) \left[\sum_{n=-\infty}^{+\infty} x_2[n] e^{-j(w-q)n} \right] dq$$

The Multiplication Property

- The bracketed summation is the discrete-time Fourier transform

$$X_2(e^{j(w-q)}) = \left[\sum_{n=-\infty}^{+\infty} x_2[n] e^{-j(w-q)n} \right]$$

and we have

$$Y(e^{jw}) = \frac{1}{2p} \int_{2p} X_1(e^{jq}) X_2(e^{j(w-q)}) dq$$

This corresponds to a periodic convolution of $X_1(e^{jw})$ and $X_2(e^{jw})$ and the integral is evaluated over any interval of length $2p$

Summary of Fourier Series and Transform Properties

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ continuous frequency aperiodic in frequency	$x[n] = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

Systems Characterized by Linear Constant-Coefficient Difference Equations

- An LTI system with input $x[n]$ and output $y[n]$ is described by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The frequency response of the system, $H(e^{jw})$, can be determined by applying the Fourier transform to both sides of the difference equation and using the linearity and time-shifting operations

$$\sum_{k=0}^{N} a_k e^{-jkw} Y(e^{jw}) = \sum_{k=0}^{M} b_k e^{-jkw} X(e^{jw})$$

where $X(e^{jw}) \overset{F}{\longleftrightarrow} x[n]$ and $Y(e^{jw}) \overset{F}{\longleftrightarrow} y[n]$

Systems Characterized by Linear Constant-Coefficient Difference Equations

- Solving $H(e^{jw})$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}$$

- The frequency response, $H(e^{jw})$, is a ratio of polynomials in the variable e^{-jw}
- The coefficients a_k and b_k are the same as in the difference equation

Example 5.18: First Order IIR Filter

- Consider the first order recursive or infinite impulse response (IIR) filter

$$y[n] - ay[n-1] = x[n], \quad \text{with } |a| < 1$$

- The frequency response of this system is

$$H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

- The impulse response is calculated earlier: $h[n] = a^n u[n]$

Example 5.19: Second Order IIR Filter

- The difference equation is $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$
- The frequency response is $H(e^{jw}) = \dfrac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}}$
- Factoring the denominator cascade of two 1st order sections $H(e^{jw}) = \dfrac{2}{\left(1 - \frac{1}{2}e^{-jw}\right)\left(1 - \frac{1}{4}e^{-jw}\right)}$
- Partial fraction expansion parallel interconnection of two 1st order sections $H(e^{jw}) = \dfrac{4}{1 - \frac{1}{2}e^{-jw}} - \dfrac{2}{1 - \frac{1}{4}e^{-jw}}$
- The impulse response is $h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$