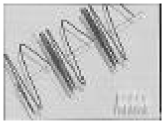
Fourier Series Representation of Continuous-Time Periodic Signals



Fourier Series Representation

- Focus on the representation of continuous-time and discrete-time periodic signals referred to as Fourier series
- Powerful and important tools for analyzing, designing, and understanding signals and LTI systems

The response of LTI systems to complex exponentials

Representation of signals as linear combinations of basic signals that have the following properties:

1) The set of basic signals can be used to construct a broad and useful class of signals
2) The response of an LTI system to each signal should be simple enough in structure to provide us with the convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals

Both of these properties are provided by the set of complex exponential signals in CT and DT

The response of LTI systems to complex exponentials

- The response of an LTI system to a complex exponential is the same complex exponential with only a change in amplitude:
  - Continuous-time: e^{sT} -> $H(s)\, e^{sT}$
  - Discrete-time: z^{n} -> $H(z)\, z^{n}$

  where the complex amplitude factor $H(s)$ or $H(z)$ will be a function of the complex variable s or z

The response of LTI systems to complex exponentials

- A signal for which the system output is a constant times the input is referred to as an ***eigenfunction*** of the system, and the amplitude value is referred to as the ***eigenvalue*** of the system
- This property for complex exponentials can be shown using:
  - The impulse response and
  - The convolution

Continuous-Time Systems

- For an input $x(t)=e^{st}$ the convolution integral gives:

$$y(t) = \int_{-\infty}^{+\infty} h(t)x(t-t)\,dt = \int_{-\infty}^{+\infty} h(t)e^{s(t-t)}dt$$

$$= \int_{-\infty}^{+\infty} h(t)e^{st}e^{-st}\,dt = e^{st}\int_{-\infty}^{+\infty} h(t)e^{-st}\,dt$$

$$= x(t)\int_{-\infty}^{+\infty} h(t)e^{-st}\,dt$$

Continuous-Time Systems

- A complex exponential $x(t)=e^{st}$ is now the eigenfunction of the LTI system with impulse response $h(t)$:

$$y(t) = x(t) \int_{-\infty}^{+\infty} h(t)e^{-st}dt = H(s)e^{st}$$

$$where \quad H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

$H(s)$ is the transfer function or system function representing the system behavior in the s-domain

Discrete-Time Systems

- For an input $x[n]=z^n$ the convolution sum gives:

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$$y[n] = z^n H(z) = x[n]H(z) \;, \quad where H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

$H(z)$ is the z-transform of the unit impulse response

$H(z)$ describes the system behavior in the z-domain

Linear Combination of Signals

- Let $x(t)$ correspond to a linear combination of complex exponentials:

$$x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}$$

- From the eigenfunction property the response to each term:

$$a_1e^{s_1t} \rightarrow a_1H(s_1)e^{s_1t}$$
$$a_2e^{s_2t} \rightarrow a_2H(s_2)e^{s_2t}$$
$$a_3e^{s_3t} \rightarrow a_3H(s_3)e^{s_3t}$$

Linear Combination of Signals

- From the superposition property the response to the sum is the sum of the responses:

$$y(t) = a_1H(s_1)e^{s_1t} + a_2H(s_2)e^{s_2t} + a_3H(s_3)e^{s_3t}$$

- The representation of signals as a linear combination of complex exponentials leads to a convenient expression for the response of an LTI system

$$Input: \quad x(t) = \sum_k a_k e^{s_kt}$$

$$Output: \quad y(t) = \sum_k a_k H(s_k)e^{s_kt}$$

Linear Combination of Signals

- In a similar way, for discrete-time systems the response of a linear combination of complex exponentials is the linear combination of individual responses:

$$Input: \quad x[n] = \sum_k a_k z_k^n$$

$$Output: \quad y[n] = \sum_k a_k H(z_k) z_k^n$$

The decomposition of more general signals in terms of eigenfunctions is the basis for frequency domain representation and analysis of LTI systems

Fourier Series Representation of Continuous-Time Periodic Signals

Linear Combinations of Harmonically Related Complex Exponentials

- A signal is periodic if for some value of T:

$$x(t) = x(t+T)\,, \quad for \quad all \quad t$$

- The fundamental period of $x(t)$ is the minimum positive, nonzero value of T for which the above is satisfied;
- The value $w_0 = 2p/T$ is referred to as the ***fundamental frequency***

Linear Combinations of Harmonically Related Complex Exponentials

- Basic periodic signals
  - Sinusoidal signal : $x(t) = \cos w_0 t$
  - Complex exponential: $x(t) = e^{jw_0 t}$
- Both of these signals are periodic with fundamental frequency w_0 and fundamental period of $T=2\pi/\omega_0$

Harmonically Related Complex Exponentials

$$f_k(t) = e^{jkw_0 t} = e^{jk(2p/T)t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- Each of these signals has a fundamental frequency that is a multiple of w_0
- Each is periodic with period T
- For $|k| \geq 2$, the fundamental period of $f_k(t)$ is a fraction of T

Linear Combination of Harmonically Related Complex Exponentials

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t}$$

- $x(t)$ is periodic with period T
- The term for $k=0$ is a constant
- The terms for $k=+1$ and $k=-1$ both have fundamental frequency equal to ω_0 and referred to as the ***fundamental components*** or the ***first harmonic components***
- The two terms for $k=+2$ and $k=-2$ are periodic with half the period of the fundamental components and referred to as the ***second harmonic components***

Example 3.2

- Construction of the signal $x(t)$ as linear combination of harmonically related sinusoidal signals
- Periodic signal: $x(t) = \sum_{k=-3}^{+3} a_k e^{jk2pt}$

$$where \quad a_0 = 1, \quad a_1 = a_{-1} = \frac{1}{4}, \quad a_2 = a_{-2} = \frac{1}{2} \quad a_3 = a_{-3} = \frac{1}{3}$$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2pt} + e^{-j2pt}\right) + \frac{1}{2}\left(e^{j4pt} + e^{-j4pt}\right) + \frac{1}{3}\left(e^{j6pt} + e^{-j6pt}\right)$$

$$x(t) = 1 + \frac{1}{2}\cos(2pt) + \cos(4pt) + \frac{2}{3}\cos(6pt)$$

Fourier Series Representation

- In general, the components for $k=+N$ and $k=-N$ are also periodic with a fraction of the period of the fundamental components and are referred to as the ***Nth harmonic components***

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t}$$

=> ***Fourier Series representation***

Determination of the Fourier Series Representation of a CT Periodid Signal

- Multiplying both sides and integrating gives:

$$x(t)e^{-jnw_0t} = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} e^{-jnw_0t}$$

$$\int_0^T x(t)e^{-jnw_0t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} e^{-jnw_0t} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0t} dt \right]$$

Determination of the Fourier Series Representation of a CT Periodid Signal

$$\int_0^T x(t)e^{-jnw_0t}dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)w_0t} dt \right]$$

$$\int_0^T e^{j(k-n)w_0t}dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

- The expression for determining the coefficients a_n is:

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jnw_0t}dt$$

Fourier Series Representation

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2p/T)t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2p/T)t}dt$$

- The set of coefficients $\{a_k\}$ are called the ***Fourier series coefficients*** or ***spectral coefficients*** of $x(t)$
- These complex coefficients measure the portion of the signal $x(t)$ that is at each harmonic of the fundamental component

Convergence of the Fourier Series

- Let us approximate a given periodic signal $x(t)$ by a linear combination of a finite number of harmonically related complex exponentials

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jkw_0t}$$

- Let $e_N(t)$ denote the approximation error

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jkw_0t}$$

Convergence of the Fourier Series

- Quantitative measure for the goodness of the approximation is defined by the energy in the error over one period

$$E_N = \int_T |e_N(t)|^2 dt$$

- It can be shown that the particular choice for coefficients that minimize the energy in the error is

$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0t}dt$$

Properties of the CT Fourier Series

$x(t)$, $y(t)$ — Periodic with period T and fundamental frequency $w_0 = 2p/T$ — a_k, b_k

Linearity:	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting:	$x(t - t_0)$	$a_k e^{-jkw_0t_0}$
Frequency shifting:	$x(t)e^{jMw_0t}$	a_{k-M}
Periodic convolution:	$\int_T x(t)y(t-t)dt$	$Ta_k b_k$
Multiplication:	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$

Fourier Series Representation of Discrete-Time Periodic Signals

The Fourier series representation of a discrete-time periodic signal is a ***finite*** series, as opposed to the infinite series representation required for continuous-time periodic signals

Linear Combinations of Harmonically Related Complex Exponentials

- A discrete-time signal is periodic with period N if

$$x[n] = x[n+N]$$

- The ***fundamental period*** is the smallest positive integer for which the above equation holds
- $w_0 = 2p/N$ is the ***fundamental frequency***

Linear Combinations of Harmonically Related Complex Exponentials

- A set of all DT complex exponential signals that are periodic with period N is given by

$$f_k[n] = e^{jkw_0 n} = e^{jk(2p/N)n}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- There are only N distinct signals in the above set due to the fact that DT complex exponentials which differ in frequency by a multiple of $2p$ are identical

$$f_k[n] = f_{k+rN}[n]$$

Linear Combinations of Harmonically Related Complex Exponentials

$$f_k[n] = f_{k+rN}[n]$$

- When k is changed by any integer multiple of N, the identical sequence is generated
- This differs from the situation in continuous-time in which the signals $f_k(t)$ are all different from one another

Linear Combinations of Complex Exponentials

$$x[n] = \sum_k a_k f_k[n] = \sum_k a_k e^{jkw_0 n} = \sum_k a_k e^{jk(2p/N)n}$$

- Since $f_k[n]$ are distinct only over a range on N successive values of k, the summation need only include terms over this range
- Expressing the limits of summation as $k = \langle N \rangle$

$$x[n] = \sum_{k=\langle N\rangle} a_k f_k[n] = \sum_{k=\langle N\rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2p/N)n}$$

- This is referred to as the discrete-time Fourier series and the coefficients a_k as the Fourier series coefficients

Discrete-Time Fourier Series

- Multiplying both sides and summing over N terms:

$$x[n]e^{-jr(2p/N)n} = \sum_{k=\langle N\rangle} a_k e^{j(k-r)(2p/N)n}$$

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2p/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2p/N)n}$$

$$= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2p/N)n}$$

Discrete-Time Fourier Series

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2p/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2p/N)n}$$

$$\sum_{n=\langle N\rangle} e^{jk(2p/N)n} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

- Now, the expression for determining the coefficients a_n is:

$$a_n = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-jr(2p/N)n}$$

Discrete Fourier Series Representation

- The synthesis and analysis equations for the discrete-time Fourier series is given by the following pair of equations:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jkw_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2p/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-jkw_0 n} = \frac{1}{N} \sum_{n=\langle N\rangle} x[n]e^{-jk(2p/N)n}$$

- The set of coefficients $\{a_k\}$ are called the discrete-time ***Fourier series coefficients*** or ***spectral coefficients*** of $x[n]$

Discrete Fourier Series Representation

$$x[n] = \sum_{k=\langle N\rangle} a_k f_k[n]$$

- If we take k in the range from 0 to N-1, we have

$$x[n] = a_0 f_0[n] + a_1 f_1[n] + \ldots + a_{N-1} f_{N-1}[n]$$

- Similarly, if k ranges from 1 to N, we have

$$x[n] = a_1 f_1[n] + a_2 f_2[n] + \ldots + a_N f_N[n]$$

- From $f_k[n] = f_{k+rN}[n]$ we have $f_0[n] = f_N[n]$
- Thus, we conclude that $a_0 = a_N$

Discrete Fourier Series Representation

- Letting k range over any set of N consecutive integers we conclude that

$$a_k = a_{k+N}$$

- The values of Fourier coefficients a_k repeat periodically with period N

Since there are only N distinct complex exponentials that are periodic with period N, the discrete-time Fourier series representation is a finite series with N terms

Properties of the DT Fourier Series

$x[n]$, $y[n]$: Periodic with period N and fundamental frequency $w_0 = 2p/N$ — a_k, b_k: Periodic with period N

Linearity:	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting:	$x[n-n_0]$	$a_k e^{-jk(2p/N)n_0}$
Frequency shifting:	$x[n]e^{jM(2p/N)n}$	a_{k-M}
Periodic convolution:	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication:	$x[n]y[n]$	$\sum_{r=\langle N\rangle} a_l b_{k-l}$

Fourier Series and Linear Time-Invariant Systems

Fourier Series and LTI Systems

- Fourier series representation can be used to construct any periodic signal in discrete-time and essentially all periodic continuous-time signals of practical importance
- The response of an LTI system to a linear combination of complex exponentials take a simple form

Response of an LTI System

In continuous-time:

$$x(t) = e^{st}; \quad y(t) = H(s)e^{st} \quad where \quad H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

In discrete-time:

$$x[n] = z^n; \quad y[n] = H(z)z^n \quad where \quad H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}$$

When s and z are general complex numbers, $H(s)$ and $H(z)$ are referred to as ***system functions***

Frequency Response

- With Re{s}=0, i.e., $s=jw$, and consequently the input $x(t)=e^{st}= e^{jwt}$ is the complex exponential at frequency w
- The system function $H(jw)$ as a function of w is given by

$$H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$$

$H(jw)$ is the ***frequency response*** of the system in continuous-time

Frequency Response

- In discrete-time, we focus on values of z for which $|z|=1$, so that $z= e^{jw}$ and $z^n= e^{jwn}$
- The system function $H(z)$ for z of the form $z= e^{jw}$ is given by

$$H(e^{jw}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jwn}$$

$H(z)$ is the ***frequency response*** of the system in discrete-time

Frequency Response

- The response of an LTI system to a complex exponential signal of the form e^{jwt} or e^{jwn} is simple to express in terms of the frequency response
- In discrete-time:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2p/N)n}$$

$$y[n] = \sum_{k=\langle N\rangle} a_k H\left(e^{j2pk/N}\right) e^{jk(2p/N)n}$$

The effect of the LTI system is to modify individually each of the Fourier coefficients of the input by multiplying it with the value of the frequency response

Filtering

Need to change the relative amplitudes of the frequency components in a signal or eliminate some frequency components entirely

=> FILTERING PROCESS

Filtering operations

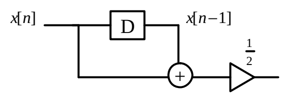
- ***Frequency-shaping filters*** are linear time-invariant systems that change the shape of the spectrum
- ***Frequency-selective filters*** are designed to pass some frequencies and significantly attenuate or eliminate others
- Fourier series coefficients of the output of an LTI system are those of the input multiplied by the frequency response of the system

Filtering operations

- Filtering can be conveniently accomplished through the use of LTI systems with an appropriately chosen frequency response
- Frequency-domain methods provide us with the ideal tools to examine this important class of applications
- Examples:
  - Frequency shaping filters: Equalizer structures
  - Frequency selective filters: Differentiating filters

Example: Image Filtering

Example: Two-Point Averaging Filter



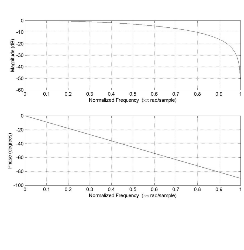
$y[n] = \frac{1}{2}(x[n] + x[n-1])$

- Impulse response: $h[n] = \frac{1}{2}(d[n] + d[n-1])$
- Frequency response:

$$H(e^{jw}) = \frac{1}{2}\left[1 + e^{-jw}\right] = e^{-jw/2}\cos(w/2)$$

$$\left|H(e^{jw})\right| = \cos(w/2), \quad and \quad \arg\left(H(e^{jw})\right) = -w/2$$

Example: Two-Point Averaging Filter



- $|H(e^{jw})|$ is large for frequencies near w=0 and decreases as w approaches p or $-p$
- Higher frequencies are attenuated more than lower ones
- The phase response is linear
- Discrete-time frequency response is periodic with period 2p

Example: Two-Point Averaging Filter

- Consider a constant input, i.e., a zero-frequency complex exponential

  $x[n] = Ke^{j0n} = K$

  The output is

  $y[n] = H(e^{j0n})Ke^{j0n} = [e^{-j0/2}\cos(0/2)]Ke^{j0n} = K = x[n]$
- If the input is the high-frequency signal

  $x[n] = Ke^{jpn} = K(-1)^n$

  The output is

  $y[n] = H(e^{pn})Ke^{pn} = [e^{-jp/2}\cos(p/2)]Ke^{pn} = 0$
- The system separates out the long-term constant value of a signal from its high-frequency fluctuations

Frequency-Selective Filters

- A class of filters specifically intended to accurately or approximately select some bands of frequencies and reject others
- Examples:
  - Removing noise in certain bands
  - Communication systems; channel separation

Frequency-Selective Filters

- Lowpass filter
  - Passes low frequencies, i.e., frequencies around $w=0$ and attenuates or rejects higher frequencies
- Highpass filter
  - Passes high frequencies and attenuates or rejects lower frequencies
- Bandpass filter
  - Passes a band of frequencies and attenuates or rejects frequencies both higher and lower than those in the band that is passed
- Bandstop filter
  - Attenuates or rejects a band of frequencies and passes frequencies both higher and lower than those in the band that is rejected

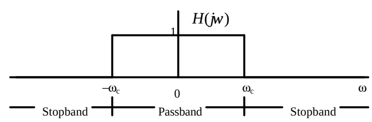
Frequency-Selective Filters

- Cutoff frequencies
  - The frequencies defining the boundaries between frequencies that are passed and frequencies that are rejected, i.e., frequencies in the ***passband*** and in the ***stopband***
- Notch filter
  - A bandstop filter which rejects a specific frequency and passes all other frequencies
- Multiband filter
  - A filter that has several passbands and stopbands
- Comb filter
  - A multiband filter in which passbands and/or stopbands are (usually) equally spaced in frequency

Ideal Frequency-Selective Filters

- ***Ideal lowpass filter*** with cutoff frequency w_c

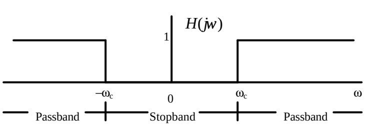
$$H(jw) = \begin{cases} 1, & |w| \le w_c \\ 0, & |w| > w_c \end{cases}$$



Ideal Frequency-Selective Filters

- ***Ideal highpass filter*** with cutoff frequency w_c

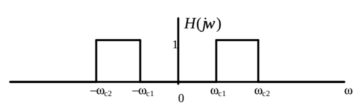
$$H(jw) = \begin{cases} 0, & |w| < w_c \\ 1, & |w| \ge w_c \end{cases}$$



Ideal Frequency-Selective Filters

- ***Ideal bandpass filter*** with cutoff frequencies w_{c1} and w_{c2}

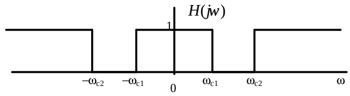
$$H(jw) = \begin{cases} 1, & w_{c1} \le |w| \le w_{c2} \\ 0, & elsewhere \end{cases}$$



Ideal Frequency-Selective Filters

- ***Ideal bandstop filter*** with cutoff frequencies w_{c1} and w_{c2}

$$H(jw) = \begin{cases} 0, & w_{c1} \le |w| \le w_{c2} \\ 1, & elsewhere \end{cases}$$

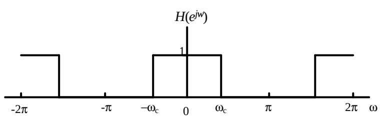


Ideal Frequency-Selective Filters

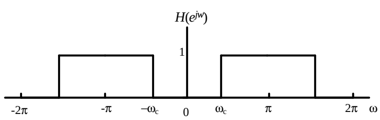
- Each of these ***ideal continuous-time filters*** is symmetric about $w=0$
- There are two passbands for the highpass and bandpass filters and three passbands for the bandstop filter
- ***Ideal discrete-time filters*** frequency-selective filters are defined in the similar way
- For discrete-time filters the frequency response is periodic with period 2π, with
  - Low frequencies near even multiples of π
  - High frequencies near odd multiples of π

Ideal Discrete-Time Frequency-Selective Filters

- Lowpass

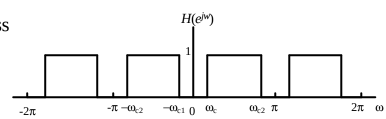


- Highpass

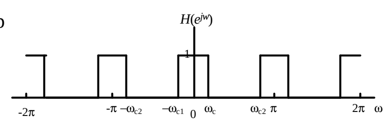


Ideal Discrete-Time Frequency-Selective Filters

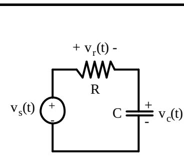
- Bandpass



- Bandstop



First-Order RC Lowpass Filter



$$RC\frac{dv_C(t)}{dt} + v_C(t) = v_s(t)$$

$$Input\ \ voltage: \ v_s(t) = e^{jwt}; \qquad Output\ \ voltage: \ v_C(t) = H(jw)e^{jwt}$$

$$RC\frac{d}{dt}\left[H(jw)e^{jwt}\right] + H(jw)e^{jwt} = e^{jwt}$$

$$RCjwH(jw)e^{jwt} + H(jw)e^{jwt} = e^{jwt}$$

$$RCjwH(jw) + H(jw) = 1 \quad \Rightarrow \quad \boxed{H(jw) = \frac{1}{1+RCjw}}$$

Transfer Function of the First-Order RC Lowpass Filter

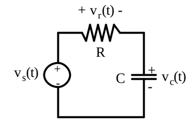
$$\boxed{H(jw) = \frac{1}{1+RCjw}}$$

- For frequencies near $w=0$, $|H(jw)|$ is close to 1
- For larger values of w, $|H(jw)|$ is considerably smaller
- $|H(jw)|$ approaches zero when w approaches infinity

Transfer Function of the First-Order RC Lowpass Filter

- Impulse response: $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$
- Step response: $s(t) = \left[1 - e^{-t/RC}\right] u(t)$
- Trade-offs in filter design:
  - Narrow passband requirement: Large *RC*
  - Fast step response: Small *RC*

First-Order RC Highpass Filter



- The output is now the voltage across the resistor

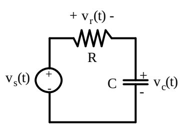
$$v_C(t) = v_s(t) - v_R(t)$$

$$i(t) = C\frac{dv_C(t)}{dt} = C\frac{d}{dt}\left(v_s(t) - v_R(t)\right)$$

$$v_R(t) = Ri(t) = RC\left[\frac{dv_s(t)}{dt} - \frac{dv_R(t)}{dt}\right]$$

$$RC\frac{dv_R(t)}{dt} + v_R(t) = RC\frac{dv_s(t)}{dt}$$

First-Order RC Highpass Filter



$$RC\frac{dv_R(t)}{dt} + v_R(t) = RC\frac{dv_s(t)}{dt}$$

Input voltage: $v_s(t) = e^{jwt}$; *Output voltage*: $v_R(t) = G(jw)e^{jwt}$

$$RC\frac{d}{dt}\left[G(jw)e^{jwt}\right] + G(jw)e^{jwt} = RC\frac{d}{dt}e^{jwt}$$

$$RCjwG(jw)e^{jwt} + G(jw)e^{jwt} = RCjwe^{jwt}$$

$$RCjwG(jw) + G(jw) = RCjw \quad\Rightarrow\quad G(jw) = \frac{jwRC}{1 + jwRC}$$

Discrete-Time Filters Described by Constant Coefficient Difference Equations

- Discrete-time LTI systems described by difference equations can be either

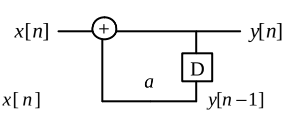
  Recursive and have an infinite impulse response (***IIR systems***)

  or

  Nonrecursive and have a finite impulse response (***FIR systems***)

- IIR systems are direct counterparts of continuous-time systems described by differential equations

First-Order Recursive Discrete-Time Filters



Difference equation:

$$y[n] - ay[n-1] = x[n]$$

Input: $x[n] = e^{jwn}$; *Output*: $y[n] = H(e^{jw})e^{jwn}$

$$H(e^{jw})e^{jwn} - aH(e^{jw})e^{jw(n-1)} = e^{jwn}$$

$$\left[1 - ae^{-jw}\right] H(e^{jw})e^{jwn} = e^{jwn} \quad\Rightarrow\quad H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

Transfer Function of the First-Order Recursive Discrete-Time Filter

$$H(e^{jw}) = \frac{1}{1 - ae^{-jw}}$$

- Parameter *a* controls the behavior of the filter
- For *a* positive => ***Lowpass filter***

  a controls the rate of attenuation at low frequencies, *w*=0
- For *a* negative => ***Highpass filter***

  a controls the rate of attenuation at high frequencies, *w*=*p*

Transfer Function of the First-Order Recursive Discrete-Time Filter

- Impulse response: $h[n] = a^n \, u[n]$
- Step response: $s[n] = u[n] * h[n] = \frac{1-a^{n+1}}{1-a} \, u[n]$
- $|a|$ controls the speed with which the impulse and step responses approach their long-term values,

  With faster responses for smaller values of $|a|$, and hence for <u>broader</u> passbands
- For $|a|<1$ the system is stable, i.e., $h[n]$ is absolutely summable

Nonrecursive Discrete-Time Filters

- General form of an FIR nonrecursive difference equation

$$y[n] = \sum_{k=-N}^{M} b_k \, x[n-k]$$

- The output is the ***weighted average*** of the $(N+M+1)$ values of $x[n]$ from $x[n-M]$ through $x[n+N]$ with the weights given by coefficients b_k.
- Such a filter is often called a ***moving-average filter***, where the output $y[n]$ for any n, e.g. for n_0, is an average of values of $x[n]$ in the vicinity of n_0

Three-Point Moving-Average Filter

- Difference equation

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

- The impulse response

$$h[n] = \frac{1}{3}\left(d[n-1] + d[n] + d[n+1]\right)$$

- The frequency response

$$H(e^{jw}) = \frac{1}{3}\left(e^{-jw} + 1 + e^{jw}\right) \quad = \frac{1}{3}\left(1 + 2\cos w\right)$$

Causal Three-Point Moving-Average Filter

- Difference equation

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

- The impulse response

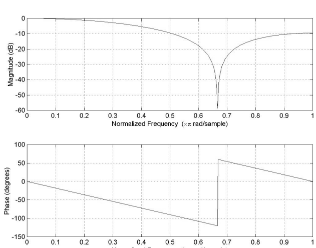
$$h[n] = \frac{1}{3}\left(d[n] + d[n-1] + d[n-2]\right)$$

- The frequency response

$$H(e^{jw}) = \frac{1}{3}\left(1 + e^{-jw} + e^{-j2w}\right) \quad = e^{-jw}\,\frac{1}{3}\left(e^{jw} + 1 + e^{-jw}\right)$$

$$= \frac{1}{3} e^{-jw}\left(1 + 2\cos w\right)$$

Three-Point Moving-Average Filter

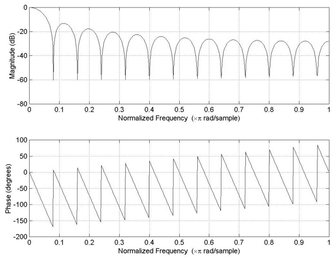


General Moving-Average Filter

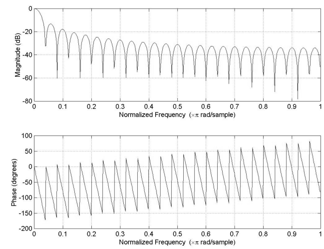
- Difference equation $y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$
- The impulse response is a rectangular pulse, i.e., $h[n]=1/(N+M+1)$ for $-N \le n \le M$ and $h[n]=0$ otherwise
- The frequency response

$$H(e^{jw}) = \frac{1}{N+M+1} e^{jw[(N-M)/2]} \, \frac{\sin[w(N+M+1)/2]}{\sin(w/2)}$$

Moving-Average Filter with N+M+1=25



Moving-Average Filter with N+M+1=50



Differentiating Nonrecursive Filter

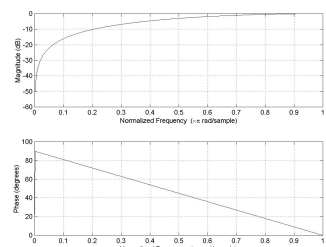
- Consider the difference equation

$$y[n] = \frac{(x[n] - x[n-1])}{2}$$

- For input signals that vary greatly from sample to sample, the value of $y[n]$ is large
- The impulse response $h[n] = \frac{1}{2}(d[n] - d[n-1])$
- The frequency response

$$H(e^{jw}) = \frac{1}{2}\left(1 - e^{-jw}\right) = je^{-jw/2}\sin(w/2)$$

First-Order Nonrecursive Highpass Filter



Nonrecursive Discrete-Time Filters

- The impulse response of a nonrecursive FIR filter is of finite length
- The impulse response is, thus, always absolutely summable for any $h[n]=b_n$

=> ***FIR filters are always stable***