2 Analyzing Self-Organization in the SOM

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The SOM algorithm despite the simplicity of its implementation has shown itself to be particularly resistant to a general analysis of its self-organizing ability. For the most part theoretical analyses of the self-organizing property have been confined to the one dimensional case, that is a one dimensional input with a one-dimensional neuron grid. One of the reasons for this limitation is that it is only in this particular case that an organized state has been, so far, rigorously defined. Despite the robustness of the SOM which has been used successfully in many different application areas, very little is know from theory what conditions are sufficient for it to self-organize, and under what conditions it cannot organize. This project is concerned with extending already existing proofs of self-organization in the SOM, in a general way, such that sufficient conditions for self-organization to occur become apparent. To explain more specifically it is necessary to introduce some notation for the SOM. As so far the project has dealt with the one dimensional case, only the one dimensional SOM is described. The input $x \in \mathbb{R}$ is considered a random variable with probability distribution P. At each time iteration t a winner neuron c is chosen such that

$$c(t) = \arg \min_i |x(t) - m_i(t)| \tag{12}$$

where $m_i(t), i = 1, \dots, N$ is the neuron weight value. Each neuron weight is then updated as follows,

$$m_i(t+1) = m_i(t) + \alpha(t)h(|i - c(t)|)(x(t) - m_i(t)). \tag{13}$$

The gain $\alpha(t)$, with $0 \leq \alpha(t) \leq 1$ during the training phase is normally a decreasing function with time. The function $h(|i - v(t)|)$ with $0 \leq h(|i - c(t)|) \leq 1$ is referred to as the *neighborhood* and $h(j)$ decreases with increasing j. In what follows $h(|i - c(t)|)$ will be written as $h(i, c(t))$. Generally h is defined as,

$$\begin{aligned} h(i,i) &= 1 \\ h(i, i \pm W) &= h_m > 0 \\ h(i,j) &= 0, \ |i-j| > W \\ h(i,j) &\leq h(i,k), \ |i-j| > |i-k| \end{aligned} \tag{14}$$

2.1 A Method for Analyzing Self-Organization

In the one dimensional case the organized configuration D of the neuron weights is *absorbing*,

$$D = \{\mathbf{M} : x_1 < x_2 < \ldots < x_N\} \bigcup \{\mathbf{M} : x_1 > x_2 > \ldots > x_N\} \tag{15}$$

and in Cottrell and Fort [2] it has been shown that from any initial condition where $m_i \neq m_j, i \neq j$, $W = 1$, and a uniform P that the weights will almost surely converge to D. This result was further generalized in Erwin et al [3], Bouton and

Pàges [4], Fort and Pàges [5], Flanagan [6], [7] and Sadeghi [8]. All of the latter consider $\mathbf{M}(t) = (m_1(t), m_2(t), \ldots,)$ as a Markov process defined on the common probability space $(\Psi, \mathcal{F}, \pi)$, and to prove self-organization it is shown that $\exists\, T < \infty$ and $\delta > 0$ for which

$$\pi_{\mathbf{M}(0)}(\{\psi \in \Psi \ : \ \tau_D \leq T\}) \ \geq \delta \tag{16}$$

or that the probability $\pi_{\mathbf{M}(0)}$, of finding sets of samples ψ in the sample space Ψ which take the neuron weights $\mathbf{M}$ from any initial condition $\mathbf{M}(0)$ to the organized configuration in a finite time τ_D is non zero. In [2], [3], [4], [5] and [8] either a uniform P or a diffuse P has been assumed. The generalization of these results is limited by the existence of situations where the inability to define a winner neuron can lead to the instability of the organized configuration. An example of when this may occur is $m_i(t) = m_j(t), i \neq j$ and

$$i, j \ = \ \arg \min_{1 \leq k \leq N} |x(t) - m_k(t)| \tag{17}$$

In [8] a modified version of the winner selection criterion of equation (12) to overcome this problem is presented along with a general analysis of the one dimensional SOM. This approach however is not generalizable to the higher dimensional case. In [6], [7] a different approach has been taken which avoids this problem of winner definition and it can be applied to both diffuse and discrete P and requires no change to the original SOM algorithm. The only restriction is that the neighborhood function $h(j)$ be assumed strictly monotonic decreasing with increasing j, that is

$$h(i,j) \leq h(i,k) - \phi \text{ for } |i-j| > |i-k|, \ \phi > 0 \tag{18}$$

In [6] it was shown that when $N \leq W$, for self-organization of the weights, the requirements on P are that its support contains a *skeleton structure* of two intervals, with $\int dP(x) > 0$ for each interval and that each interval be separated from the other by a certain minimum distance defined in terms of parameters of the map. The condition on P can be used both for discrete and diffuse P and this proof has already been easily extended to higher dimensional SOMs [6], which suggests the general framework of the proof developed in this project is not restricted to the one dimensional case. Define the *order*, n of an SOM as

$$n = \begin{cases} [\frac{N}{W}] + 1, & N \bmod W \neq 0 \\ \frac{N}{W}, & N \bmod W = 0 \end{cases} \tag{19}$$

and in this project the results of [6] (i.e. $n = 1$), and [7] (i.e. $n = 2$) are generalized, for the one dimensional case, for any $n \geq 1$. In other words general conditions that the SOM and support of P must satisfy are described and it is then proven that these conditions are sufficient for self-organization of the neuron weights for any $n \geq 1$ and any initial state of the neuron weights.

2.2 A Structure for Self-Organization

In the course of the project a structure has been defined, which if it exists in the support of P then self-organization can be shown. This structure $\mathcal{A}_n$, associated with an SOM of degree n, is defined in terms of two structures $\mathcal{A}_{n-1}$, separated by a certain minimum distance which depend on parameters of the SOM. Hence the structure $\mathcal{A}_n$ is defined recursively from 2^n basic structures $\mathcal{A}_0$, which is quite basically an interval on the line. Given this structure $\mathcal{A}_n$ with an SOM of degree n, then assuming that

$$\int_{\mathcal{A}_0} dP(x) > 0 \tag{20}$$

then the following theorem can be stated and proved.

Theorem 1 *For any initial, finite* $\mathbf{M}(0)$ *and the structure* $\mathcal{A}_n$ *such that* $N \leq nW$ *with*

$$\int_{\mathcal{A}_0} dP(x) \;\geq\; \epsilon, \qquad \epsilon > 0 \tag{21}$$

for every $\mathcal{A}_0$ *interval in* $\mathcal{A}_n$*, then* $\exists\, T < \infty$ *and* $\delta > 0$ *for which*

$$\pi_{\mathbf{M}(0)}(\{\psi \in \Psi \;:\; \tau_D \leq T\}) \;\geq \delta \tag{22}$$

where τ_D *is the first entry time of* $\mathbf{M}$ *into* D*.*

The recursive nature of the structure $\mathcal{A}_n$ leads to a proof by induction of the theorem. Throughout the proof three basic principles which apply at the level of the neuron weight updates are used, they are referred to as, convergence, order preservation and one step organization, and as well as being used in the one dimensional case, similar principles have been applied to higher dimensional SOMs.

2.3 Conclusion

By defining a special structure $\mathcal{A}_n$ on the support of P a general proof of self-organization in a one dimensional SOM has been given. The proof itself is theoretical, which raises many interesting questions concerning the implications of the proof in a practical situation. The conditions as determined are sufficient for self-organization to occur, but are they necessary in a practical situation, if not, how close are they to being necessary ? This question is very difficult from a theoretical point of view, given that the system being dealt with is stochastic. An estimation only of the importance of the conditions can be obtained from simulations.
Another interesting point of the structure $\mathcal{A}_n$ is the fact that it is self-similar, that is it looks the same on a large or small scale. In [9] the existence of a $1/f$ spectrum for the update of the neuron weights during training was shown by simulation. This is interesting in that there is a more general class of non linear systems which are referred to as *emergent*, they are usually associated with some form of self-similarity and a $1/f$ spectrum of some form. The significance if any of this relative to the SOM remains to be seen.

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