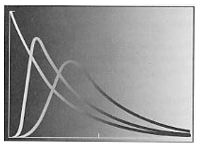
Linear Time-Invariant Systems

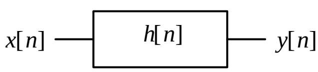


Basic Properties LTI Systems

- The Commutative Property
- The Distributive Property
- The Associative Property

Convolution

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$



$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

The Commutative Property

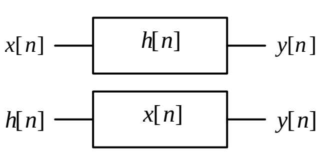
$$x[n]*h[n] = h[n]*x[n]$$

- Let $r=n-k$ or $k=n-r$; substituting to convolution sum:

$$x[n]*h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] =$$

$$\sum_{r=-\infty}^{\infty} x[n-r]h[r] = h[n]*x[n]$$

The Commutative Property



- The output of an LTI system with input $x[n]$ and unit impulse response $h[n]$ is identical to the output of an LTI system with input $h[n]$ and unit impulse response $x[n]$

The Distributive Property

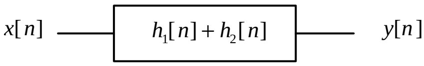
$$x[n]*(h_1[n]+h_2[n]) =$$

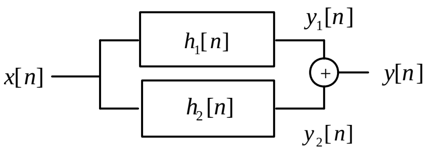
$$= x[n]*h_1[n] + x[n]*h_2[n]$$

- The distributive property has a useful interpretation in terms of system interconnections

=> ***PARALLEL INTERCONNECTION***

The Distributive Property





The Associative Property

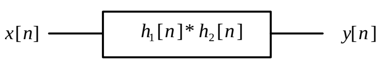
$$x[n]*\left(h_1[n]*h_2[n]\right)=$$
$$=\left(x[n]*h_1[n]\right)*h_2[n]$$

- As a consequence of associative property the following expression is unambiguous

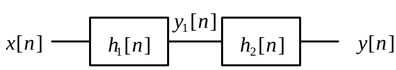
$$y[n]=x[n]*h_1[n]*h_2[n]$$

The Associative Property

$$y[n]=x[n]*\left(h_1[n]*h_2[n]\right)$$



$$y[n]=\left(x[n]*h_1[n]\right)*h_2[n]=\ y_1[n]*h_2[n]$$



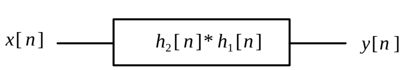
The Associative Property

- The associative property can be interpreted as

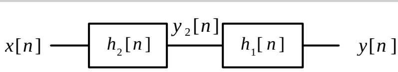
=> ***SERIES (OR CASCADE) INTERCONNECTION OF SYSTEMS***

The Associative and Commutative Property

$$y[n]=x[n]*\left(h_1[n]*h_2[n]\right)=x[n]*\left(h_2[n]*h_1[n]\right)$$



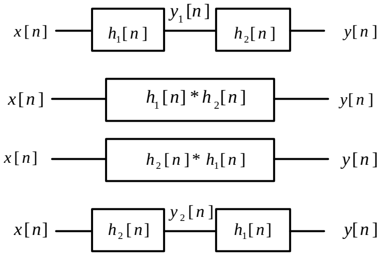
$$y[n]=\left(x[n]*h_2[n]\right)*h_1[n]=y_2[n]*h_1[n]$$



The Properties of Cascade Connection of Systems

- The order of the systems in cascade can be interchanged
- The intermediate signal values, $w_i[n]$, between the systems are different

The Cascade Connection of Systems



The Cascade Connection of Systems

- The properties of the cascade system depend on the sequential order of cascaded blocks
- The behavior of discrete-time systems with finite wordlength is sensitive to signal values, $w_i[n]$, between the blocks
- ***What is the optimal sequential order of cascaded blocks ?***

Stability for LTI Systems

Definition of stability:

A system is stable if and only if every bounded input produces a bounded output

Bounded-input bounded-output (BIBO) stability

Stability for LTI Systems

- Consider an input $x[n]$ that is bounded in magnitude

$$|x[n]| < B \text{ for all } n$$

- The output is given by the convolution sum

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right|$$

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

Stability for LTI Systems

- For bounded input $x[n-k]<B$

$$|y[n]| \le B\sum_{k=-\infty}^{\infty} |h[k]| \qquad for \quad all \quad n$$

- The output $y[n]$ is bounded if the the impulse response is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

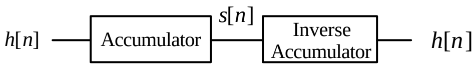
A SUFFICIENT CONDITION FOR STABILITY !

The Unit Step Response

- The unit step response is often used to characterize LTI systems

$$s[n] = u[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} u[k]h[n-k]$$
$$= \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$
$$= \sum_{k=-\infty}^{n} h[k]$$

Invertible Systems



$$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad h[n] = s[n] - s[n-1]$$

- The step response of a DT LTI system is the running sum its impulse response and the impulse response of a DT LTI system is the first difference of its step response

Causal LTI Systems

- Continuous-time systems: ***Linear constant-coefficient differential equations*** are used to describe a wide variety of systems and physical phenomena
- Discrete-time systems: ***Linear constant-coefficient difference equations*** are used to describe the sequential behavior of many different proceses

Linear Constant-Coefficient Difference Equations

- Nth-order constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The condition for initial rest:

  If $x[n]=0$ for $n < n_0$, then $y[n]=0$ for $n < n_0$

With initial rest the system is LTI and causal

Linear Constant-Coefficient Difference Equations

- Rearranging and solving for $y[n]$

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

- The output $y[n]$ at time n is expressed in terms of previous values of the input and output
- Auxiliary conditions: In order to calculate $y[n]$ we need to know $y[n-1], \ldots, y[n-N]$

Linear Constant-Coefficient Difference Equations

$$y[n] = \frac{1}{a_0}\left\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right\}$$

- ***Recursive equation:***

  The output is specified recursively in terms of the input and previous outputs

Linear Constant-Coefficient Difference Equations

- In a special case when N=0 the difference equation reduces to

$$y[n] = \left\{ \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k] \right\} = \sum_{k=0}^{M} h[k] x[n-k]$$

- ***Nonrecursive equation:***

  Previously computed output values are not needed to compute the present value of the output

  Auxiliary conditions are not needed !

Linear Constant-Coefficient Difference Equations

- The impulse response corresponding to the nonrecursive system is

$$h[n] = \begin{cases} \dfrac{b_n}{a_0}, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

The system specified by the nonrecursive equation is often called a Finite Impulse Response (FIR) system

Linear Constant-Coefficient Difference Equations

- Consider the difference equation

$$y[n] - \frac{1}{2} y[n-1] = x[n]$$

$$y[n] = x[n] + \frac{1}{2} y[n-1]$$

- The impulse response is obtained as a response to $x[n]=d[n]$

$$n=0: \quad y[0] = d[0] + \frac{1}{2} y[-1] = 1 + \frac{1}{2} y[-1] = 1$$

$$n=1: \quad y[1] = d[1] + \frac{1}{2} y[0] = 0 + \frac{1}{2} y[0] = \frac{1}{2}$$

Impulse Response of a First-Order System

- The impulse response can be written as

$$h[n] = \begin{cases} \left(\dfrac{1}{2}\right)^n, & n \ge 0 \\ 0, & n < 0 \end{cases}$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Linear Constant-Coefficient Difference Equations

- Difference equations with $N \geq 1$ are recursive and result in an impulse response of infinite length

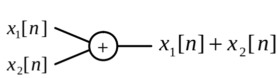
The systems specified by recursive equations are called Infinite Impulse Response (IIR) systems

- In general, recursive difference equations will be used in describing and analyzing discrete-time systems that are linear, time-invariant, and causal, and consequently the assumption of initial rest will usually be made

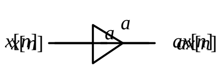
Block Diagram Representation of Systems

Basic elements:

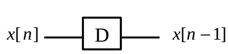
- Addition:



- Multiplication:



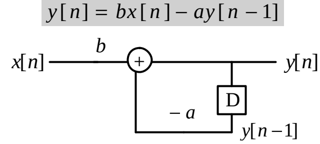
- Unit delay:



First–Order Difference Equation

$$y[n] + ay[n-1] = bx[n]$$

$$y[n] = bx[n] - ay[n-1]$$



- Block diagram representation of a causal discrete-time system (a first-order IIR digital filter)

Linear Constant-Coefficient Differential Equations

- Consider a first order differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

where $y(t)$ denotes the output of the system and $x(t)$ is the input

- Differential equations provide ***implicit*** specification of the system, i.e., the relationship between the input and output

Linear Constant-Coefficient Differential Equations

- In order to obtain an explicit solution, the differential equation must be solved
- More information is needed than that provided by the equation alone, i.e., auxiliary conditions must be specified

=> *A differential equation describes a constraint between the input and output of the system, but to characterize the system completely auxiliary conditions must be specified*

Linear Constant-Coefficient Differential Equations

- The response to an input x(t) will generally consist of the sum of
  - a *particular solution*, $y_p(t)$, to the differential equation - a signal of the same form as the input - i.e., the *forced response*, and
  - a *homogeneous solution*, $y_h(t)$, - a solution to the differential equation with the input set to zero - i.e., the *natural response*

$$y(t) = y_p(t) + y_h(t)$$

Linear Constant-Coefficient Differential Equations

Auxiliary conditions must be specified:

- Different choices of auxiliary conditions lead to different relationships between the input and output
- For the most part, the condition of initial rest will be used for systems described by differential equations, e.g., $x(t)=0$ for $t<0$, the condition for initial rest implies the initial condition $y(0)=0$

Linear Constant-Coefficient Differential Equations

- Under the condition of initial rest the system is linear time-invariant (LTI) and causal
- The condition of initial rest does not specify a zero initial condition at a fixed point of time, but rather adjusts this point in time so that the response is zero *until* the input becomes nonzero
- For example, if $x(t)=0$ for $t \leq t_0$ for a causal LTI system described a differential equation, then $y(t)=0$ for $t \leq t_0$ and the initial condition $y(t_0)=0$ would be used to solve the output for $t>t_0$

Linear Constant-Coefficient Differential Equations

- A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where the order refers to the highest derivative of $y(t)$

- In the case when $N=0$

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$y(t)$ is an explicit function of $x(t)$ and its derivatives

Linear Constant-Coefficient Differential Equations

- For $N \geq 1$, the output is specified implicitly by the input
- The solution of the equation consists of two parts:
  - a particular solution and
  - a solution of the homogeneous differential equation
- The solutions to the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

are referred to as natural responses of the system

Linear Constant-Coefficient Differential Equations

- In order to determine the input-output relationship of the system completely, auxiliary conditions must be identified
- Different choices of auxiliary conditions result in different input-output relationships
- The condition of initial rest: If $x(t)=0$ for $t \leq t_0$, it is assumed that $y(t)=0$ for $t \leq t_0$ and, therefore, the response for $t > t_0$ can be calculated from

$$y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

- Under the condition of initial rest, the system is causal and LTI