Basic Properties LTI Systems

- The Commutative Property
- The Distributive Property
- The Associative Property

**Convolution**

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[n] * h[n] \]

**The Commutative Property**

\[ x[n] * h[n] = h[n] * x[n] \]

- Let \( r = n-k \) or \( k = n-r \); substituting to convolution sum:

\[ x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{r=-\infty}^{\infty} x[n-r] h[r] = h[n] * x[n] \]

**The Distributive Property**

\[ x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n] \]

- The distributive property has a useful interpretation in terms of system interconnections

\( \Rightarrow \text{PARALLEL INTERCONNECTION} \)
The Distributive Property

\[ x[n] \cdot h_1[n] + h_2[n] = y[n] \]

The Associative Property

\[ x[n] \cdot (h_1[n] \cdot h_2[n]) = (x[n] \cdot h_1[n]) \cdot h_2[n] \]

- As a consequence of associative property the following expression is unambiguous

\[ y[n] = x[n] \cdot h_1[n] \cdot h_2[n] \]

The Associative Property

\[ y[n] = x[n] \cdot (h_1[n] \cdot h_2[n]) \]

\[ y[n] = (x[n] \cdot h_1[n]) \cdot h_2[n] = y_1[n] \cdot h_2[n] \]

\[ y[n] = x[n] \cdot h_1[n] \cdot h_2[n] \]

- The associative property can be interpreted as

\[ \Rightarrow \text{SERIES (OR CASCADE) INTERCONNECTION OF SYSTEMS} \]

The Associative and Commutative Property

\[ y[n] = x[n] \cdot (h_1[n] \cdot h_2[n]) = x[n] \cdot (h_2[n] \cdot h_1[n]) \]

\[ y[n] = (x[n] \cdot h_2[n]) \cdot h_1[n] = y_2[n] \cdot h_1[n] \]

\[ y[n] = x[n] \cdot h_1[n] \cdot h_2[n] \]

- The order of the systems in cascade can be interchanged
- The intermediate signal values, \( w[n] \), between the systems are different
The Cascade Connection of Systems

The properties of the cascade system depend on the sequential order of cascaded blocks.

• The behavior of discrete-time systems with finite wordlength is sensitive to signal values, \( w[n] \), between the blocks.

• What is the optimal sequential order of cascaded blocks?

Stability for LTI Systems

Definition of stability:

**A system is stable if and only if every bounded input produces a bounded output**

Bounded-input bounded-output (BIBO) stability

For bounded input \( s[n-k] < B \)

\[
| y[n] | \leq B \left| \sum_{k=\infty}^{\infty} h[k] x[n-k] \right| \quad \text{for all } n
\]

• The output \( y[n] \) is bounded if the impulse response is absolutely summable:

\[
\sum_{k=\infty}^{\infty} |h[k]| < \infty
\]

*A SUFFICIENT CONDITION FOR STABILITY!*

The Unit Step Response

• The unit step response is often used to characterize LTI systems:

\[
s[n] = u[n] * h[n]
= \sum_{k=-\infty}^{\infty} u[k] h[n-k]
= \sum_{k=0}^{n} h[k] u[n-k]
= \sum_{k=0}^{n} h[k]
\]
Invertible Systems

- The step response of a DT LTI system is the running sum of its impulse response and the impulse response of a DT LTI system is the first difference of its step response.

Causal LTI Systems

- Continuous-time systems: Linear constant-coefficient differential equations are used to describe a wide variety of systems and physical phenomena.
- Discrete-time systems: Linear constant-coefficient difference equations are used to describe the sequential behavior of many different processes.

Linear Constant-Coefficient Difference Equations

- Nth-order constant-coefficient difference equation:
  \[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]
  - The condition for initial rest:
    - If \( x[n]=0 \) for \( n < n_0 \), then \( y[n]=0 \) for \( n < n_0 \).
    - With initial rest the system is LTI and causal.

- The output is specified recursively in terms of the input and previous outputs.

Linear Constant-Coefficient Difference Equations

- Nonrecursive equation:
  - Previously computed output values are not needed to compute the present value of the output.
  - Auxiliary conditions are not needed!

Linear Constant-Coefficient Difference Equations

- Recursive equation:
  - The output is specified recursively in terms of the input and previous outputs.

Linear Constant-Coefficient Difference Equations

- In a special case when \( N=0 \) the difference equation reduces to:
  \[ y[n] = \frac{1}{a_0} \sum_{k=0}^{N} b_k x[n-k] \]
  - Nonrecursive equation:
    - Previously computed output values are not needed to compute the present value of the output.
    - Auxiliary conditions are not needed!
Linear Constant-Coefficient Difference Equations

- The impulse response corresponding to the nonrecursive system is

\[ h[n] = \begin{cases} \frac{b}{a}, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases} \]

The system specified by the nonrecursive equation is often called a Finite Impulse Response (FIR) system.

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Impulse Response of a First-Order System

- The impulse response can be written as

\[ h[n] = \begin{cases} \frac{1}{2}a^n, & n \geq 0 \\ 0, & n < 0 \end{cases} \]

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Block Diagram Representation of Systems

Basic elements:
- Addition: \( x[n], y[n] \)
- Multiplication: \( a[n], b[n] \)
- Unit delay: \( x[n-1] \)

First-Order Difference Equation

\[ y[n] + ay[n-1] = bx[n] \]

\[ y[n] = bx[n] \]

First-Order Difference Equation

\[ y[n] + ay[n-1] = bx[n] \]

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Linear Constant-Coefficient Differential Equations

- Consider a first order differential equation
  \[ \frac{dy(t)}{dt} + 2y(t) = x(t) \]
  where \( y(t) \) denotes the output of the system and \( x(t) \) is the input.

- Differential equations provide **implicit** specification of the system, i.e., the relationship between the input and output.

- In order to obtain an explicit solution, the differential equation must be solved.
- More information is needed than that provided by the equation alone, i.e., auxiliary conditions must be specified.

- A differential equation describes a constraint between the input and output of the system, but to characterize the system completely auxiliary conditions must be specified.

- The response to an input \( x(t) \) will generally consist of the sum of:
  - A **particular solution**, \( y_p(t) \), to the differential equation - a signal of the same form as the input - i.e., the **forced response**.
  - A **homogeneous solution**, \( y_h(t) \), - a solution to the differential equation with the input set to zero - i.e., the **natural response**.

\[
y(t) = y_p(t) + y_h(t)
\]

- Auxiliary conditions must be specified:
  - Different choices of auxiliary conditions lead to different relationships between the input and output.
  - For the most part, the condition of initial rest will be used for systems described by differential equations, e.g., \( x(t) = 0 \) for \( t < 0 \), the condition for initial rest implies the initial condition \( y(0) = 0 \).

- Under the condition of initial rest the system is linear time-invariant (LTI) and causal.
- The condition of initial rest does not specify a zero initial condition at a fixed point of time, but rather adjusts this point in time so that the response is zero until the input becomes nonzero.
- For example, if \( x(t) = 0 \) for \( t < t_0 \) for a causal LTI system described a differential equation, then \( y(t) = 0 \) for \( t < t_0 \), and the initial condition \( y(t_0) = 0 \) would be used to solve the output for \( t > t_0 \).

- A general \( N \)th-order linear constant-coefficient differential equation is given by
  \[
  \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} + \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} = 0
  \]
  where the order refers to the highest derivative of \( y(t) \).
- In the case when \( N > 0 \)
  \[
  y(t) = \frac{1}{a_N} \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}
  \]
  \( y(t) \) is an explicit function of \( x(t) \) and its derivatives.
Linear Constant-Coefficient Differential Equations

• For \( N \geq 1 \), the output is specified implicitly by the input
• The solution of the equation consists of two parts:
  – a particular solution and
  – a solution of the homogeneous differential equation
• The solutions to the homogeneous differential equation
  \[ \sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0 \]
  are referred to as natural responses of the system

In order to determine the input-output relationship of the system completely, auxiliary conditions must be identified.

Different choices of auxiliary conditions result in different input-output relationships.

The condition of initial rest: If \( y(t)=0 \) for \( t < t_0 \), it is assumed that \( y(t)=0 \) for \( t < t_0 \) and, therefore, the response for \( t \geq t_0 \) can be calculated from

\[ y(t_0) = \frac{dy(t_0)}{dt} = \ldots = \frac{d^{N-1} y(t_0)}{dt^{N-1}} = 0 \]

Under the condition of initial rest, the system is causal and LTI.