Tik-61.261 Principles of Neural Computing

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1. The McCulloch-Pitts perceptrons can be used to perform numerous logical tasks. Neurons are assumed to have two binary input signals, x_1 and x_2, and a constant bias signal which are combined into an input vector as follows: $\mathbf{x} = [x_1, x_2, -1]^T$, $x_1, x_2 \in \{0, 1\}$. The output of the neuron is given by

$$y = \begin{cases} 1, & \text{if } \mathbf{w}^T\mathbf{x} > 0 \\ 0, & \text{if } \mathbf{w}^T\mathbf{x} \le 0 \end{cases}$$

   where $\mathbf{w}$ is an adjustable weight vector. Demonstrate the implementation of the following binary logic functions with a single neuron:

   (a) A

   (b) not B

   (c) A or B

   (d) A and B

   (e) A nor B

   (f) A nand B

   (g) A xor B.

   What is the value of weight vector in each case?

2. A single perceptron is used for a classification task, and its weight vector $\mathbf{w}$ is updated iteratively in the following way:

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \alpha(y - y')\mathbf{x}$$

   where $\mathbf{x}$ is the input signal, $y' = \mathrm{sgn}(\mathbf{w}^T\mathbf{x}) = \pm 1$ is the output of the neuron, and $y = \pm 1$ is the correct class. Parameter α is a positive learning rate. How does the weight vector $\mathbf{w}$ evolve from its initial value $\mathbf{w}(0) = [1, 1]^T$, when the above updating rule is applied with $\alpha = 0.4$, and we have the following samples from classes $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\begin{aligned} \mathcal{C}_1 &: \ \{[2, 1]^T\}, \\ \mathcal{C}_2 &: \ \{[0, 1]^T, [-1, 1]^T\} \end{aligned}$$

3. Suppose that in the signal-flow graph of the perceptron illustrated in Figure 1 the hard limiter is replaced by the sigmoidal linearity:

$$\varphi(v) = \tanh(\frac{v}{2})$$

   where v is the induced local field. The classification decisions made by the perceptron are defined as follows:

   Observation vector $\mathbf{x}$ *belongs to class* $\mathcal{C}_1$ *if the output* $y > \theta$ *where* θ *is a threshold; otherwise,* $\mathbf{x}$ *belongs to class* $\mathcal{C}_2$

   Show that the decision boundary so constructed is a hyperplane.

4. Two pattern classes, $\mathcal{C}_1$ and $\mathcal{C}_2$, are assumed to have Gaussian distributions which are centered around points $\mu_1 = [-2, -2]^T$ and $\mu_2 = [2, 2]^T$ and have the following covariance matrixes:

$$\Sigma_1 = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \Sigma_2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}.$$

Plot the distributions and determine the optimal Bayesian decision surface for $\alpha = 3$ and $\alpha = 1$. In both cases, assume that the prior probabilities of the classes are equal, the costs associated with correct classifications are zero, and the costs associated with misclassifications are equal.

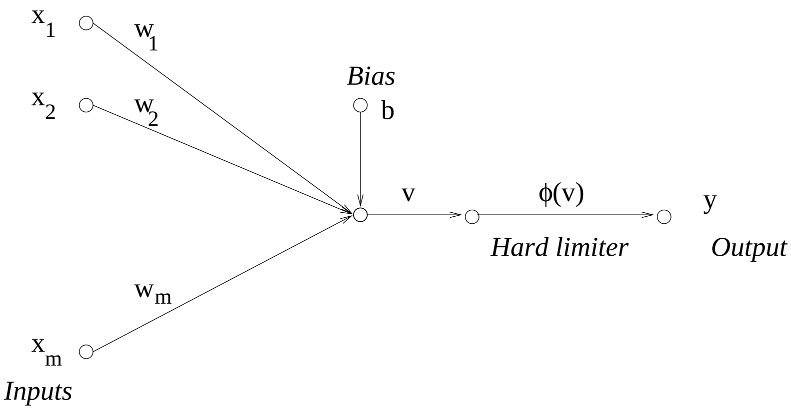

Figure 1: The signal-flow graph of the perceptron.