T-61.5040 Oppivat mallit ja menetelmät
T-61.5040 Learning Models and Methods
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Problem 1.

You have observed vectors $y_1, \ldots, y_n$ which are independent and normally distributed with

$$p(y_i|\theta, V) = N(y_i|\theta, V) \propto \exp\left(-\frac{1}{2}(y_i - \theta)^T V^{-1}(y_i - \theta)\right).$$

The covariance matrix V is known. Compute the posterior of θ assuming that the prior of θ is $N(\theta_0, V_0)$.

Problem 2.

What is the solution of the previous problem when you have a single scalar observation y? Write the posterior mean as

$$\mathrm{E}(\theta|y) = \theta_0 + (y - \theta_0)C$$

and compute the multiplier C.
Check what happens when the prior variance σ_0^2 and the likelihood variance σ^2 differ significantly.

Problem 3.

Assume that the variance σ^2 of a Normal distribution $N(\mu, \sigma^2)$ is unknown, but the mean μ is known.

i) Compute the posterior of σ^2 having observed one value $y \sim N(\mu, \sigma^2)$. Use an Inverse Gamma prior

$$p(\sigma^2) = \mathrm{IG}(\sigma^2|a, b) \propto (\sigma^2)^{-(a+1)} \exp(-b/\sigma^2).$$

ii) Write the posterior as an Inverse Gamma distribution $\mathrm{IG}(\sigma^2|a', b')$. What are the parameters a' and b'?

Problem 4.

The Jeffreys' prior for a scalar parameter θ is defined as

$$p(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta)$ is the Fisher information matrix

$$I(\theta) = \mathrm{E}\Big(\big[\frac{\partial}{\partial\theta} \log p(y|\theta)\big]^2\Big).$$

The expectation is computed over $p(y|\theta)$.

i) Compute the Jeffreys' prior for the model $p(y|\theta) = N(y|\theta, \sigma^2)$.
Here θ is an example of a *location parameter.*

ii) Compute the Jeffreys' prior for the model $p(y|\theta) = N(y|\mu, \theta^2)$.
Here θ is an example of a *scaling parameter.* You can use the result that if $y \sim N(\mu, \sigma^2)$, the fourth central moment $\mathrm{E}([y-\mu]^4)=3\sigma^4$.