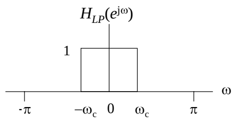
7 LTI Discrete-Time Systems in the Transform Domain

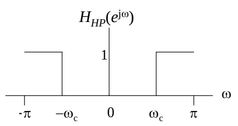
Introduction

- Time-domain classification of a digital filter transfer function based on the length of its impulse response leads to the ***finite impulse response*** (FIR) and the ***infinite impulse response*** (IIR) transfer functions
- In the frequency-domain, four types of ideal filters with frequency-selective frequency responses are usually defined
- The definition is based on the shape of the magnitude function $|H(e^{j\omega})|$

Ideal Filters

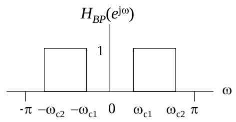


Lowpass filter:
- Passband $0 \le \omega \le \omega_c$
- Stopband $\omega_c < \omega \le \pi$

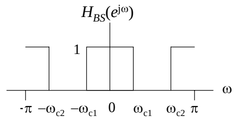


Highpass filter:
- Stopband $0 \le \omega < \omega_c$
- Passband $\omega_c \le \omega \le \pi$

Ideal Filters



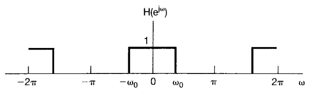
Bandpass filter:
- Passband $\omega_{c1} \le \omega \le \omega_{c2}$
- Stopbands $0 \le \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi$



Bandstop filter:
- Passbands $0 \le \omega \le \omega_{c1}$ and $\omega_{c2} \le \omega \le \pi$
- Stopband $\omega_{c1} < \omega < \omega_{c2}$

Impulse Response of the Ideal Lowpass Filter

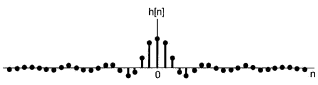
$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{+\omega_c} e^{j\omega n}d\omega = \frac{\sin \omega_c n}{\pi n}$$



(a)

- The impulse response is a sinc function

$$h[n] \ne 0, \quad \text{for } n < 0$$



(b)

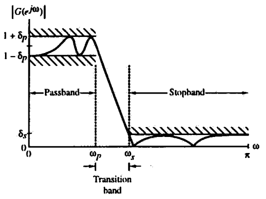
- The impulse response is not causal and its oscillatory behavior is not desired

Ideal Filter Characteristics

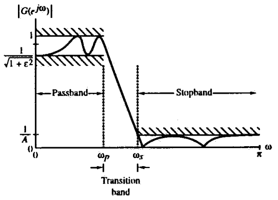
- All the ideal filters are characterized by doubly infinite, non-causal impulse responses
- As a result, ideal filters with "brick wall" frequency domain characteristics cannot be realized by an LTI with a transfer function of finite order
- In order to develop stable and realizable transfer functions, a finite ***transition band*** in introduced between the passband and stopband
- The magnitude response is allowed to vary by a specified amount both in the passband and in the stopband

Ideal Filter Characteristics

- Typical magnitude specifications for the design of lowpass filters:



(a) Magnitude specifications



(b) Normalized magnitude specifications

Bounded Real Transfer Functions

- A causal real-coefficient $H(z)$ is defined as a ***bounded real*** **(BR)** ***transfer function*** if

$$\left|H(e^{j\omega})\right| \le 1, \quad \text{for all values of } \omega$$

- Note that any stable real-coefficient transfer function can be made into a BR function by appropriate scaling

Bounded Real Transfer Functions

- If the Fourier transforms of the input and output of a digital filter characterized by a BR transfer function $H(z)$ are $X(e^{j\omega})$ and $Y(e^{j\omega})$, respectively, then

$$\left|Y(e^{j\omega})\right|^2 \le \left|X(e^{j\omega})\right|^2$$

- In other words, for all finite-energy inputs, the output energy is less than or equal to the input energy
- This implies that a digital filter characterized by a BR transfer function can be viewed as a passive structure

Lossless Bounded Real Transfer Functions

- If the BR condition is satisfied with the equality sign

$$\left|H(e^{j\omega})\right| = 1, \quad \text{for all values of } \omega$$

the output energy is equal to the input energy

- Such a digital filter is a ***lossless*** system
- A causal stable real-coefficient transfer function $H(z)$ with frequency response $H(e^{j\omega})$ of unity magnitude is called a ***lossless bounded real*** **(LBR)** ***transfer function***

Allpass Transfer Functions

<u>Definition</u>:

An IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$|A(e^{j\omega})|^2 = 1, \quad \text{for all } \omega$$

is called an ***allpass transfer function***

- Now an M^{th} order causal real-coefficient allpass transfer function is of the form

$$A_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}}$$

Allpass Transfer Functions

- If we denote the denominator polynomial of the allpass function $A_M(z)$ as $D_M(z)$

$$D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

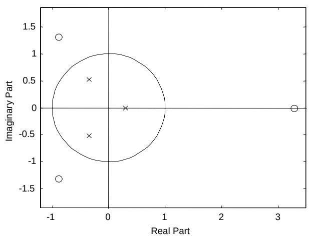
- It follows that $A_M(z)$ can be written as

$$A_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Note from above that if $z = re^{j\varphi}$ is a pole of $A_M(z)$ then it has a zero at $z = (1/r)e^{-j\varphi}$, i.e., $D_M(z^{-1})$ is a ***mirror-image polynomial*** of $D_M(z)$, and vice versa

Allpass Transfer Functions

- Poles and zeros of a real coefficient allpass function exhibit ***mirror-image symmetry*** in the z-plane
- Example: The pole-zero diagram of a third order allpass function



Allpass Transfer Functions

- To show that $A_M(e^{j\omega})$ is indeed equal to one for all ω, it follows from $A_M(z)$ that

$$A_M(z)=\pm\frac{z^{-M}D_M(z^{-1})}{D_M(z)} \quad\Rightarrow\quad A_M(z^{-1})=\pm\frac{z^{M}D_M(z)}{D_M(z^{-1})}$$

- Therefore $A_M(z)A_M(z^{-1})=\frac{z^{-M}D_M(z^{-1})}{D_M(z)}\frac{z^{M}D_M(z)}{D_M(z^{-1})}=1$
- Hence

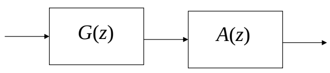
$$\left|A_M(e^{j\omega})\right|^2=A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}}=1$$

Allpass Transfer Function

A Simple Application

- A simple but often used application of an allpass filter is as a ***delay equalizer***
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

Allpass Filter as a Delay Equalizer



- Since $|A(e^{j\omega})|=1$, we have

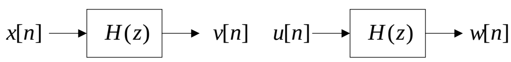
$$|G(e^{j\omega})A(e^{j\omega})|=|G(e^{j\omega})|$$

- Overall group delay is then given by the sum of the group delays of $G(z)$ and $A(z)$
- The allpass section is designed so that the overall group delay is approximately constant in the frequency range of interest

Zero-Phase Transfer Functions

- Frequency response of the filter is real and non-negative
  => *zero-phase characteristics*
- Impossible to design a causal digital filter with zero phase
- Non-real time processing of finite length, zero-phase filtering can be implemented if the causality requirement is relaxed:
  - The finite-length input data is processed by a causal real coefficient filter $H(z)$ whose output is then time-reversed and processed by the same filter once again

Zero-Phase Filtering



$$u[n]=v[\pm n], \quad y[n]=w[\pm n]$$

- Let $v[-n]=u[n]$

$$\Rightarrow \quad V(e^{j\omega})=H(e^{j\omega})X(e^{j\omega}), \quad W(e^{j\omega})=H(e^{j\omega})U(e^{j\omega})$$
$$U(e^{j\omega})=V^*(e^{j\omega}), \quad Y(e^{j\omega})=W^*(e^{j\omega})$$

$$\Rightarrow \quad Y(e^{j\omega})=W^*(e^{j\omega})=H^*(e^{j\omega})U^*(e^{j\omega})$$
$$=H^*(e^{j\omega})V(e^{j\omega})=H^*(e^{j\omega})H(e^{j\omega})X(e^{j\omega})$$
$$=|H(e^{j\omega})|^2\,X(e^{j\omega})$$

Linear-Phase Transfer Function

- In the case of a causal LTI system with nonzero phase response, the phase distortion can be avoided by allowing the output to be a delayed version of the input

$$y[n] = x[n - D]$$

- The Fourier transform gives $Y(e^{j\omega}) = e^{-j\omega D} X(e^{j\omega})$
- The frequency response is

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega D}$$

Linear-Phase Transfer Functions

- The frequency response has a unity magnitude and a linear phase with a group delay of D samples at all frequencies

$$|H(e^{j\omega})| = 1, \quad \tau(\omega) = D$$

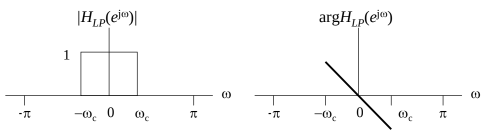
- The output of this filter to an input $x[n]=Ae^{j\omega n}$ is the given by

$$y[n] = Ae^{-j\omega D} e^{j\omega n} = Ae^{-j\omega(n-D)}$$

- The output is the delayed version of the input

Linear-Phase Transfer Functions

- If we desire to pass input signal components in a certain frequency range undistorted both in magnitude and phase, then the transfer function should exhibit a unity magnitude response and linear-phase in the band of interest



Minimum-Phase and Maximum-Phase Transfer Functions

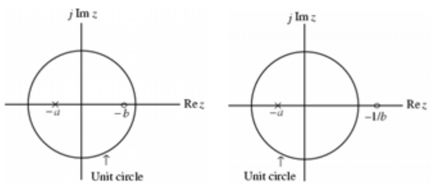
- Consider the two 1st order transfer functions:

$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

- Both transfer functions have a pole inside the unit circle at the same location $z = -a$ and are stable
- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$, whereas, the zero of $H_2(z)$ is at $z = -1/b$ situated in a mirror-image symmetry

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions



$H_1(z)$ $H_2(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- However, both transfer functions have an identical magnitude function as

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) = 1$$

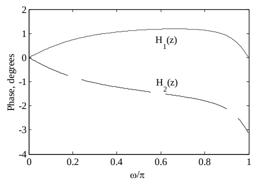
- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for a=0.8 and b=-0.5



Minimum-Phase and Maximum-Phase Transfer Functions

- From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$
- Generalizing the above result, we can show that a causal stable transfer function with all zeros outside the unit circle has an excess phase compared to a causal transfer function with identical magnitude but having all zeros inside the unit circle

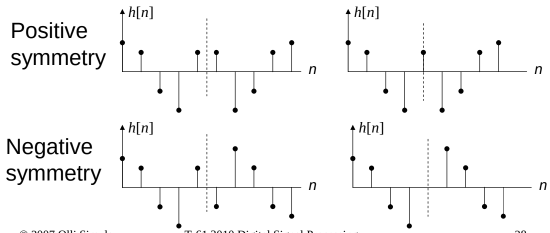
Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros inside the unit circle is called a ***minimum-phase transfer function***
- A causal stable transfer function with all zeros outside the unit circle is called a ***maximum-phase transfer function***
- Any nonminimum-phase transfer function can be expressed as the product of a minimum-phase transfer function and a stable allpass transfer function

FIR Filters with Linear Phase Response

- Necessary condition for linear-phase in FIR filters is the symmetry of the impulse response

$$h[n] = \pm h[N-n]$$



Linear Phase Response

Type 1: Symmetric impulse response with odd length

$$h[n] = h[N-n], \quad 0 \le n \le N$$

Assume that N=6:

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} + h[6]z^{-6}$$

$$= h[0](1+z^{-6}) + h[1](z^{-1}+z^{-5}) + h[2](z^{-2}+z^{-4}) + h[3]z^{-3}$$

$$= z^{-3}\left\{h[0](z^{3}+z^{-3}) + h[1](z^{2}+z^{-2}) + h[2](z+z^{-1}) + h[3]\right\}$$

$$H(e^{j\omega}) = e^{-j3\omega}\underbrace{\left\{2h[0]\cos(3\omega) + 2h[1]\cos(2\omega) + 2h[2]\cos(\omega) + h[3]\right\}}$$

$$\Rightarrow \theta(\omega) = -3\omega \qquad \left|H(e^{j\omega})\right|; \quad \text{(real)}$$

Linear Phase Response

Type 1: Symmetric impulse response with odd length

$$H(e^{j\omega}) = e^{-jN\omega/2}\left\{\sum_{n=0}^{N/2} a[n]\cos(\omega n)\right\}$$

$$\text{where } a[0] = h\left[\frac{N}{2}\right], \quad a[n] = 2h\left[\frac{N}{2} - n\right], \quad 1 \le n \le \frac{N}{2}$$

Type 2: Symmetric impulse response with even length

$$H(e^{j\omega}) = e^{-jN\omega/2}\left\{\sum_{n=1}^{(N+1)/2} b[n]\cos(\omega(n - ½))\right\}$$

$$\text{where } b[n] = 2h\left[\frac{N+1}{2} - n\right], \quad 1 \le n \le \frac{N+1}{2}$$

Linear Phase Response

Type 3: Antisymmetric impulse response with odd length

$$H(e^{j\omega}) = e^{-jN\omega/2} e^{j\pi/2} \left\{ \sum_{n=1}^{N/2} c[n] \sin(\omega n) \right\}$$

$$\text{where } c[n] = 2h\left[\frac{N}{2} - n\right], \quad 1 \le n \le \frac{N}{2}$$

Type 4: Antisymmetric impulse response with even length

$$H(e^{j\omega}) = e^{-jN\omega/2} e^{j\pi/2} \left\{ \sum_{n=1}^{(N+1)/2} d[n] \sin\left(\omega(n - ½)\right) \right\}$$

$$\text{where } d[n] = 2h\left[\frac{N+1}{2} - n\right], \quad 1 \le n \le \frac{N+1}{2}$$

Zero Locations of Linear-Phase FIR Filters

- Consider an FIR filter with symmetric impulse response

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n} = \sum_{n=0}^{N} h[N-n] z^{-n}$$

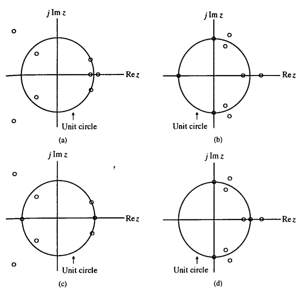
- Substituting $m=N-n$ results in

$$H(z) = \sum_{m=0}^{N} h[m] z^{-N+m} = z^{-N} \sum_{n=0}^{N} h[m] z^{m} = z^{-N} H(z^{-1})$$

- The similar relation holds for antisymmetric impulse response, i.e. $h[n]=-h[N-n]$
- Thus, if $z=\xi_0$ is a zero of $H(z)$ then so is $z=1/\xi_0$

Zero Locations of Linear-Phase FIR Filters

- For an FIR filter with real impulse response, the zeros occur in complex conjugate pairs
- Thus, the zeros of linear-phase FIR filters occur in a set of four zeros at $z_{1,2}=re^{\pm j\varphi}$ and $z_{3,4}=(1/r)e^{\pm j\varphi}$

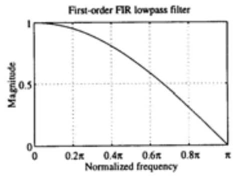


Simple FIR Digital Lowpass Filter

- The simplest moving average filter with $M=2$ has a transfer function

$$H_0(z) = \frac{1}{2}\left(1 + z^{-1}\right)$$

with a frequency response $H_0(e^{j\omega}) = e^{-j\omega/2} \cos(\omega/2)$



- Lowpass FIR filter
- Zero at $z = -1$

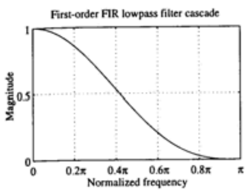
$$\max\{|H_0(e^{j\omega})|\} = 1, \quad \text{at } \omega = 0$$

$$\min\{|H_0(e^{j\omega})|\} = 0, \quad \text{at } \omega = \pi$$

Higher Order FIR Lowpass Filters

- A cascade of three first order simple FIR filter sections results in improved lowpass frequency response

$$H(z) = \left[\frac{1}{2}\left(1 + z^{-1}\right)\right]^3$$



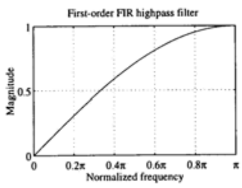
- Lowpass FIR filter
- Three zeros at $z = -1$

Simple FIR Highpass Filter

- The simplest highpass filter is obtained from the lowpass filter by replacing z with $-z$ resulting in a transfer function

$$H_1(z) = \frac{1}{2}\left(1 - z^{-1}\right)$$

with a frequency response $H_1(e^{j\omega}) = e^{-j\omega/2} j \sin(\omega/2)$



- Highpass FIR filter
- Zero at $z = 1$

$$\max\{|H_0(e^{j\omega})|\} = 1, \quad \text{at } \omega = \pi$$

$$\min\{|H_0(e^{j\omega})|\} = 0, \quad \text{at } \omega = 0$$

Simple IIR Lowpass Filter

- A first order lowpass IIR digital filter has a transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\,\frac{1+z^{-1}}{1-\alpha\, z^{-1}}, \quad \text{where } |\alpha| < 1$$

- The squared magnitude of the frequency response is

$$\left|H_{LP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

- The filter is a lowpass filter:

$$\max\{|H_{LP}(e^{j\omega})|\} = 1, \quad \text{at } \omega = 0$$
$$\min\{|H_{LP}(e^{j\omega})|\} = 0, \quad \text{at } \omega = \pi$$

Simple IIR Lowpass Filter

- The 3-dB cutoff frequency is obtained by setting

$$\left|H_{LP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)} = \frac{1}{2}$$

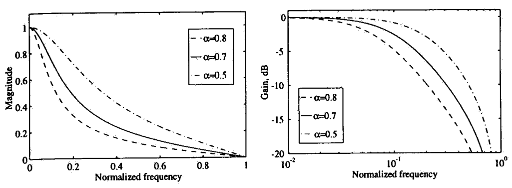
which yields $\cos\omega_c = \dfrac{2\alpha}{1+\alpha^2}$

- The parameter α can now be expressed as

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

Simple IIR Lowpass Filter

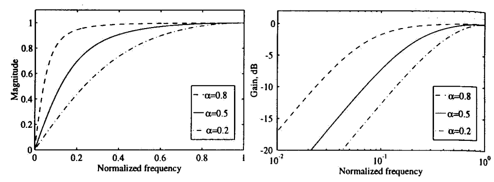
- The plots of the magnitude response and the gain in dB scale with different values of α are shown



Simple IIR Highpass Filter

- A first order highpass IIR digital filter has a transfer function

$$H_{HP}(z) = \frac{1-\alpha}{2}\,\frac{1-z^{-1}}{1-\alpha\, z^{-1}}, \quad \text{where } |\alpha| < 1$$



Second-Order IIR Bandpass Filter

- A second order bandpass IIR digital filter has a transfer function

$$H_{BP}(z) = \frac{1-\alpha}{2}\,\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha\, z^{-2}}$$

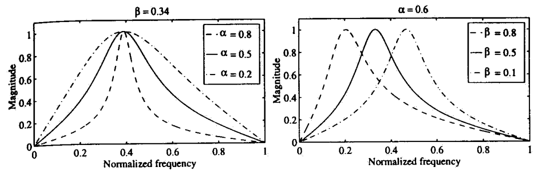
- The maximum value of unity of the magnitude response is at $\omega = \omega_0$, called the ***center frequency*** of the bandpass filter

$$\omega_0 = \cos^{-1}(\beta)$$

- The parameter α controls the sharpness of the response

Second-Order IIR Bandpass Filter

- Plots of the magnitude responses for several values of α and β are given below

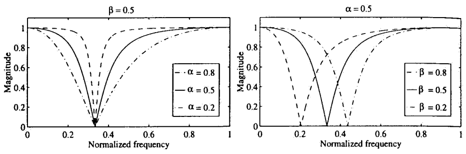


- Zeros: $1-z^{-2}=(1+z^{-1})(1-z^{-1})=0 \Rightarrow z=\underline{+}1$
- Poles: $p_{1,2}=re^{\pm j\phi}$

Second Order IIR Bandstop Filter

- A transfer function of the form

$$H_{BS}(z) = \frac{1+\alpha}{2}\,\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$



- Zeros on the unit circle at $\omega = \cos^{-1}(\beta)$

Higher Order Transfer Functions

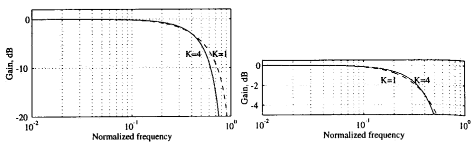
- Sharper magnitude responses can be obtained by increasing the order of the transfer function
- In general, high order digital filters are realized as cascade or parallel combinations of first and second order blocks
- Cascade form: $H(z) = \prod_{i=1}^{N} H_i(z)$
- Parallel form: $H(z) = \sum_{k=1}^{N} H_k(z)$
- Lower order sections are preferred due to their better performance with finite wordlength computation

First Order Sections in Cascade

- K first order sections in cascade:

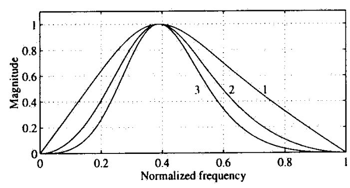
$$G_{LP}(z) = \left(\frac{1-\alpha}{2}\,\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)^{K}, \quad \text{where } |\alpha| < 1$$

$$\left|G_{LP}(e^{j\omega})\right|^{2} = \left[\frac{(1-\alpha)^{2}(1+\cos\omega)}{2(1+\alpha^{2}-2\alpha\cos\omega)}\right]^{K}$$



Bandpass Sections in Cascade

- Cascading second order bandpass filter sections sharpens the response
- Example: One, two, or three sections in cascade:



Comb Filters

- Multiple passbands and stopbands required
- Frequency response is periodic function of ω with period $2\pi/M$
- Lowpass type: $G_0(z) = H_0(z^M) = \frac{1}{2}\left(1+z^{-M}\right)$
- Notch filter with M notch frequencies at $\omega = (2k+1)\pi/M,\ k = 0, 1, ..., M-1$
- Highpass type: $G_1(z) = H_1(z^M) = \frac{1}{2}\left(1-z^{-M}\right)$
- Notch filter with M notch frequencies at $\omega = 2k\pi/M,\ k = 0, 1, ..., M-1$

Comb Filters

- The substitution of z^M in place of z can be implemented by replacing each delay by M delays
- Magnitude responses of FIR comb filters generated from the lowpass filter $H_0(z)=½(1+z^{-1})$ and the highpass filter $H_1(z)=½(1-z^{-1})$ with $M=5$