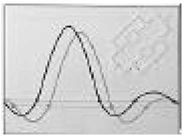
The Continuous-Time Fourier Transform



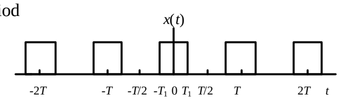
The Continuous-Time Fourier Transform

- We shall discuss signals that are not periodic
- Aperiodic signals in continuous time are represented by the Fourier transform
- An aperiodic signal can be viewed as a periodic signal with an infinite period
- As the period becomes infinite, the frequency components form a continuum and the Fourier series becomes an integral

Representation of Aperiodic Signals

- Revisiting the Fourier series:
  - Consider the continuous-time periodic square wave, i.e., over one period

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$



- The Fourier series coefficients, a_k, are:

$$a_k = \frac{2\sin(kw_0 T_1)}{kw_0 T} \quad \text{where} \quad w_0 = 2p/T$$

Representation of Aperiodic Signals

- An alternative representation is as samples of the envelope function

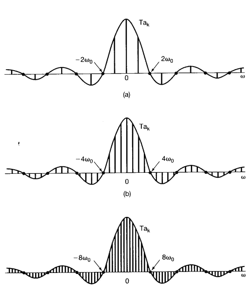
$$Ta_k = \frac{2\sin(wT_1)}{w}\bigg|_{w = kw_0}$$

- With w thought as a continuous variable the function $(2\sin wT_1)/w$ represents the envelope of Ta_k, and a_k are the samples
- For fixed T_1 the envelope Ta_k is independent of T

Representation of Aperiodic Signals

- The Fourier series coefficients and their envelope for the periodic square wave for several values of T (T_1 fixed)

  a) $T = 4T_1$
  b) $T = 8T_1$
  c) $T = 16T_1$



Representation of Aperiodic Signals

- As T increases, or equivalently, as the fundamental frequency $w_0 = 2p/T$ decreases, the envelope is sampled with a closer and closer spacing
- As T becomes arbitrarily large, the periodic square wave approaches a rectangular pulse
- Also the Fourier series coefficients , multiplied by T, become more and more closely spaced samples

=> ***The Fourier series coefficients approach the envelope***

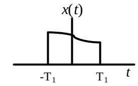
Representation of Aperiodic Signals

- The basic idea behind the development is that an aperiodic signal is thought as a limit of a periodic signal as the period becomes arbitrarily large and the limiting behavior of the Fourier series is considered

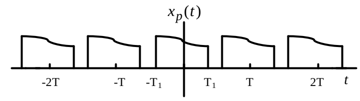
Representation of Aperiodic Signals

- Consider a signal $x(t)$ of ***finite duration***

$$x(t) = 0, \quad |t| > T_1$$



- We construct a periodic signal $x_p(t)$ for which $x(t)$ is one period



- As we choose the period T to be larger $x_p(t)$ is identical to $x(t)$ over a longer interval, and as T approaches infinity, $x_p(t)$ is equal to $x(t)$ for any finite value of t

Representation of Aperiodic Signals

- Let us examine the effect on the Fourier series representation o f $x_p(t)$
- ***Fourier series:***

$$x_p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) e^{-jkw_0 t} dt$$

where $w_0 = 2p/T$.

- Since $x_p(t)=x(t)$ for $|t|<T/2$, and also, since $x(t)=0$ outside this interval

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jkw_0 t} dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt$$

Representation of Aperiodic Signals

- Defining the envelope $X(jw)$ of Ta_k as

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jkw_0 t} dt$$

we have for the coefficients a_k

$$a_k = \frac{1}{T} X(jkw_0)$$

- We can now express $x_p(t)$ in terms of $X(jw)$ as

$$x_p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jkw_0) e^{jkw_0 t}$$

Representation of Aperiodic Signals

- Equivalently, since $2p/T=w_0$

$$x_p(t) = \frac{1}{2p} \sum_{k=-\infty}^{+\infty} X(jkw_0) e^{jkw_0 t} w_0$$

- As T approaches infinity, $x_p(t) \Rightarrow x(t)$ and the above equation becomes the representation of $x(t)$
- Furthermore, $w_0 \Rightarrow 0$ as T approaches infinity, and the summation passes to an integral

The Fourier Transform Pair

$$x(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} X(jw) e^{jwt} dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t) e^{-jwt} dt$$

Fourier Transform Pair

- $X(jw)$ is ***Fourier transform*** or ***Fourier integral*** of $x(t)$, i.e., the analysis equation
- The ***inverse Fourier transform*** equation is the synthesis equation
- For aperiodic signals, the complex exponentials occur at a continuum of frequencies
- The transform $X(jw)$ of an aperiodic signal $x(t)$ is commonly referred to as the ***spectrum*** of $x(t)$

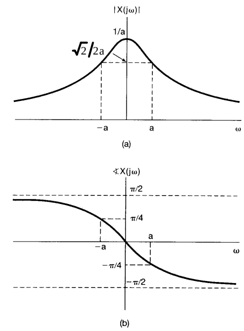
Example 4.1

- The Fourier transform of a causal complex exponential

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$X(jw) = \int_0^{\infty} e^{-at}e^{-jwt}dt$$

$$= \frac{1}{a + jw}, \quad a > 0$$



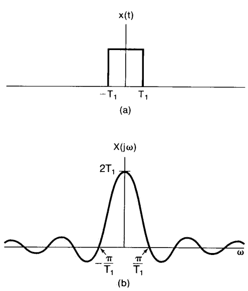
Example 4.4

- The Fourier transform of a rectangular pulse

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$$

$$X(jw) = \int_{-T_1}^{T_1} e^{-jwt}dt$$

$$= 2\frac{\sin wT_1}{w}$$



Example 4.5

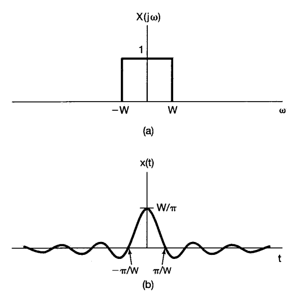
- Consider the signal with the Fourier transform

$$X(jw) = \begin{cases} 1, & |w| < W \\ 0, & |w| > W \end{cases}$$

- Using the synthesis equation

$$x(t) = \frac{1}{2p}\int_{-W}^{W} e^{jwt}dw$$

$$= \frac{\sin Wt}{pt}$$



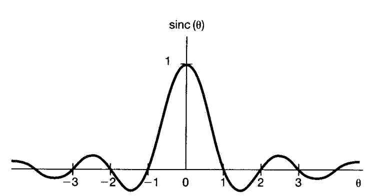
Duality Property of the Fourier Transform

- In examples 4.4 and 4.5 the Fourier transform pair consists of a function of the form $(\sin aQ)/bQ$ and a rectangular pulse
- In Example 4.4, it is the ***signal*** $x(t)$ that is a pulse, while in Example 4.5 it is the ***transform*** $X(jw)$
- This is the consequence of the ***duality property*** of the Fourier transform

Sinc functions

- A commonly used precise form of the ***sinc function*** is

$$\text{sinc}(q) = \frac{\sin pq}{pq}$$



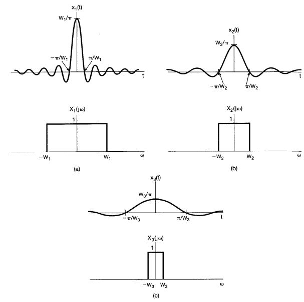
Sinc functions

Both of the signals in examples 4.4 and 4.5 can be expressed in terms of the sinc functions

$$X(jw) = 2\frac{\sin(wT_1)}{w} = 2T_1\frac{\sin\left(\frac{pwT_1}{p}\right)}{\frac{pwT_1}{p}} = 2T_1\text{sinc}\left(\frac{wT_1}{p}\right)$$

$$x(t) = \frac{\sin(Wt)}{pt} = \frac{W}{p}\frac{\sin\left(\frac{pWt}{p}\right)}{\frac{pWt}{p}} = \frac{W}{p}\text{sinc}\left(\frac{Wt}{p}\right)$$

Properties of the Square Pulse and Its Fourier Transform (Sinc function)



- As W increases, $X(jw)$ becomes broader while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of $x(t)$ becomes narrower ($|t|<p/W$)
- In the limit, $x(t)$ converges to an impulse as W -> ∞

The Fourier Transform of Periodic Signals

- We can construct a Fourier transform of a periodic signal directly from its Fourier series representation
- The transform consists of a train of impulses in the frequency domain, with the areas of the impulses proportional to the Fourier series coefficients
- Consider a signal $x(t)$ with a Fourier transform $X(jw)$ that is a single impulse of area $2p$ at $w=w_0$

$$X(jw) = 2pd(w - w_0)$$

- $x(t)$ is obtained from the inverse transform relation

$$x(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} 2p\ d(w - w_0)e^{jwt}dt = e^{jw_0t}$$

The Fourier Transform of Periodic Signals

- More generally, if $X(jw)$ is of the form of a linear combination of impulses equally spaced in frequency, i.e.,

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2p\ a_k d(w - kw_0)$$

the inverse transform relation yields

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0t}$$

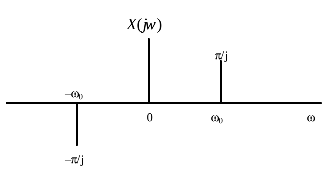
which corresponds to the Fourier series representation of a periodic signal

Example 4.7: A Sinusoidal Signal (1)

$$x(t) = \sin(w_0t) = \frac{1}{2j}\left(e^{jw_0t} - e^{-jw_0t}\right)$$

$$\text{where} \quad a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}$$

$$\text{and} \quad a_k = 0, \quad k \neq 1 \ or \ -1$$

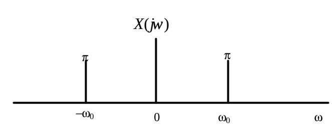


Example 4.7: A Sinusoidal Signal (2)

$$x(t) = \cos(w_0t) = \frac{1}{2}\left(e^{jw_0t} + e^{-jw_0t}\right)$$

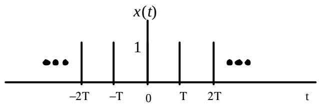
$$\text{where} \quad a_1 = a_{-1} = \frac{1}{2},$$

$$\text{and} \quad a_k = 0, \quad k \neq 1 \ or \ -1$$



Example 4.8: A Periodic Impulse Train

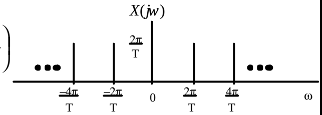
$$x(t) = \sum_{k=-\infty}^{+\infty} d\ (t - kT)$$



Fourier series coefficients:
(Calculated in Example 3.8)

$$a_k = \frac{1}{T}\int_{-T/2}^{+T/2} d\,(t)\ e^{-jkw_0 t}\,dt = \frac{1}{T}$$

$$X(jw) = \frac{2p}{T}\sum_{k=-\infty}^{+\infty} d\left(w - \frac{2pk}{T}\right)$$



Properties of the Periodic Impulse Train

- The Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2p/T$
- The inverse relationship between the time and the frequency domains:
  As the spacing between the impulses in the time domain (i.e. the period) gets longer, the spacing between the impulses in the frequency domain (i.e. the fundamental frequency) gets smaller

The result is very useful in the analysis of sampling systems

Properties of the Continuous-Time Fourier Transform

Fourier transform pairs: $x(t) \overset{F}{\longleftrightarrow} X(jw)\ ,\ \ y(t) \overset{F}{\longleftrightarrow} Y(jw)$

- Linearity: $ax(t) + by(t) \overset{F}{\longleftrightarrow} aX(jw) + bY(jw)$
- Time Shifting: $x(t - t_0) \overset{F}{\longleftrightarrow} e^{-jwt_0} X(jw)$
- Convolution property: If $h(t) \overset{F}{\longleftrightarrow} H(jw)$ then

$$y(t) = x(t) * h(t) \overset{F}{\longleftrightarrow} Y(jw) = X(jw)H(jw)$$

Convolution in the time domain corresponds to multiplication in the frequency domain

Convolution Property

- A signal $x(t)$ can be expressed as linear combination of complex exponentials

$$x(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw = \lim_{w_0 \to 0}\frac{1}{2p}\sum_{k=-\infty}^{\infty} X(jkw_0)e^{jkw_0 t}w_0$$

- Frequency response: $H(jw) = \int_{-\infty}^{+\infty} h(t)e^{-jwt}dt$
- The frequency response is defined as the Fourier transform of the impulse response h(t)
- The Fourier transform of the impulse response (at $w = kw_0$) is the complex scaling factor that the LTI system applies to eigenfunction $e^{jkw_0 t}$

Convolution Property

- From superposition:

$$\frac{1}{2p}\sum_{k=-\infty}^{\infty} X(jkw_0)e^{jkw_0 t}w_0 \ \rightarrow\ \frac{1}{2p}\sum_{k=-\infty}^{\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

- Thus, the response of a linear system to x(t) is

$$y(t) = \lim_{w_0 \to 0}\frac{1}{2p}\sum_{k=-\infty}^{\infty} X(jkw_0)H(jkw_0)e^{jkw_0 t}w_0$$

$$= \frac{1}{2p}\int_{-\infty}^{\infty} X(jw)H(jw)e^{jwt}dw$$

- Definition (synthesis equation) $y(t) = \frac{1}{2p}\int_{-\infty}^{\infty} Y(jw)e^{jwt}dt$

Duality Property

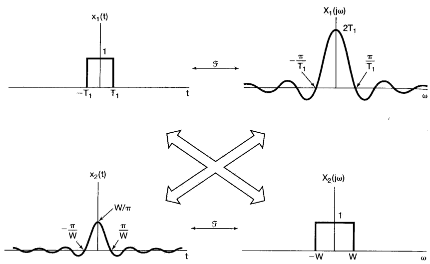
- By comparing the Fourier transform and inverse transform relations

$$x(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} X(jw)e^{jwt}dw$$

$$X(jw) = \int_{-\infty}^{+\infty} x(t)e^{-jwt}dt$$

  we notice that the equations are similar in form (but not quite identical)
- This symmetry was also noticed in the examples 4.4 and 4.5, i.e., the transform of a square pulse was the sinc function and vice versa

Duality Property



The Frequency Response

- Frequency response plays an important role in the analysis of LTI systems as does the impulse response response

  Since the impulse response $h(t)$ completely characterizes an LTI system, then so must $H(j\omega)$
- The parallel and cascade connections of LTI systems can be easily specified using the frequency responses instead of the impulse responses
- <u>Systems in parallel:</u> $h_1(t)+h_2(t) \overset{F}{\longleftrightarrow} H_1(j\omega)+H_2(j\omega)$
- <u>Systems in cascade:</u> $h_1(t)*h_2(t) \overset{F}{\longleftrightarrow} H_1(j\omega)H_2(j\omega)$

The Multiplication (or Modulation) Property

- <u>Duality:</u>

  Convolution in the time domain corresponds to multiplication in the frequency domain and

  Multiplication in the time domain corresponds to convolution in the frequency domain

$$r(t)=s(t)p(t) \overset{F}{\longleftrightarrow} R(j\omega)=\frac{1}{2\pi}[S(j\omega)*P(j\omega)]$$

- Multiplication of one signal by another can be thought of as using one signal to scale or ***modulate*** the amplitude of the other, and the operation is referred to as ***amplitude modulation***

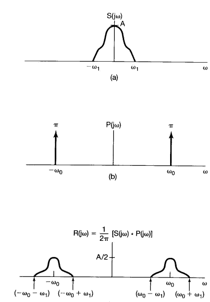
Example 4.21: Modulation

- Let $s(t)$ be a signal with the spectrum $S(j\omega)$
- Consider also a signal $p(t)$ with the Fourier transform $P(j\omega)$

$$p(t) = \cos \omega_0 t$$

$$P(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

- The product $r(t)= p(t)s(t)$ has the spectrum $R(j\omega)$



Amplitude Modulation

- By multiplying the signal $s(t)$ with a sinusoidal signal, we notice that
  - The information has been shifted to higher frequencies, i.e., to the frequency (ω_0) of the modulating signal $p(t)$
  - All the information in the original signal $s(t)$ is preserved
- This fact forms the basis for the sinusoidal amplitude modulation systems in communications
- The original signal $s(t)$ can be easily recovered from the amplitude modulated signal $r(t)$

Example 4.22: Demodulation

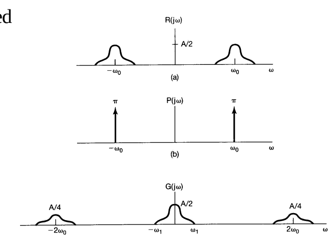
- Consider the modulated signal $r(t)$ of example 4.21 and let

$$g(t) = r(t)p(t)$$

where $p(t)$ is again

$$p(t) = \cos \omega_0 t$$

- The spectra $R(j\omega)$, $P(j\omega)$, and $G(j\omega)$ are shown on the right



- The signal $s(t)$ can be recovered from $g(t)$ using a lowpass filter

Example: Modulation and Demodulation in Signal Processing

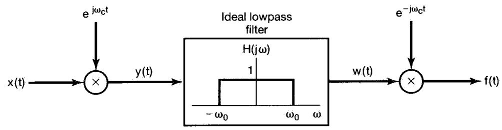


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

$$y(t) = e^{jw_c t} x(t) \qquad f(t) = e^{-jw_c t} w(t)$$

$$Y(jw) = X(j(w - w_c)) \qquad F(jw) = W(j(w + w_c))$$

Spectra of the signals

a) Spectrum of the original signal $x(t)$

b) Spectrum of the amplitude-modulated signal $y(t)$

c) Spectrum of the lowpass filtered signal $w(t)$

d) Spectrum of the demodulated signal, i.e., the output signal $f(t)$