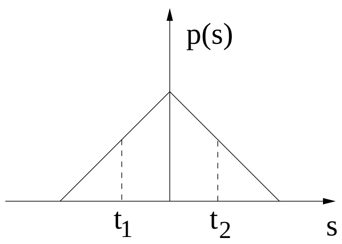
T-61.5100 Digital image processing, Exercise 9/07

1.

It is given that

$$p(s) = \begin{cases} s+1, & \text{when } -1 < s < 0 \\ 1-s, & \text{when } 0 \le s < 1 \\ 0, & \text{otherwise} \end{cases}$$



Because $p(s)$ is symmetric around some point (here it is 0), the reconstruction levels of the quantizer $t_i,\ i = 1, 2$ are also set symmetrically around 0.
Minimizing the mean-square error function is equivalent with setting the reconstruction level to its center of gravity. Let us solve t_2 from the following equation:

$$\int_0^{\infty} (s - t_2)p(s)\, ds = 0.$$

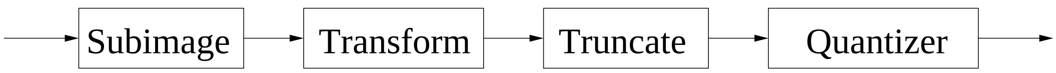
$$\int_0^{\infty} (s - t_2)p(s)\, ds = \int_0^{1} (s - t_2)(1 - s)\, ds = \int_0^{1} (-s^2 + (1 + t_2)s - t_2)\, ds = -\frac{1}{3} + \frac{1 + t_2}{2} - t_2 = 0$$

$$\Leftrightarrow \quad t_2 = 2(-\frac{1}{3} + \frac{1}{2}) = \frac{1}{3}$$

t_1 is set symmetrically to $t_1 = -1/3$.

2.

The encoder is the following:



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{y}' = \begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

a) The average truncation (mapping) error:

$$e_m^2 = E\left\{\|\mathbf{y} - \mathbf{y}'\|^2\right\} = E\left\{\sum_{i=1}^{n} (y_i - y'_i)^2\right\} = \sum_{i=1}^{n} E\left\{(y_i - y'_i)^2\right\}$$

$$= \sum_{i=1}^{m} E\{(\underbrace{y_i - y'_i}_{0})^2\} + \sum_{i=m+1}^{n} E\{(\underbrace{0 - y'_i}_{(\star)})^2\} = \sum_{i=m+1}^{n} E\left\{{y'_i}^2\right\}.$$

$(\star)$ The components that were left out were set as zeros.

b)

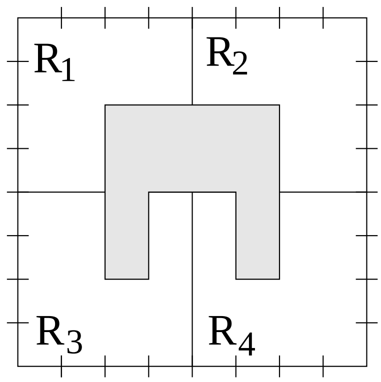
$$\begin{aligned} e_T^2 &= E\left\{\|\mathbf{v}-\mathbf{y}'\|^2\right\} = E\left\{\|(\mathbf{v}-\mathbf{y})+(\mathbf{y}-\mathbf{y}')\|^2\right\} \\ &= E\left\{\|\mathbf{v}-\mathbf{y}\|^2 + 2(\mathbf{v}-\mathbf{y})^T(\mathbf{y}-\mathbf{y}') + \|\mathbf{y}-\mathbf{y}'\|^2\right\} \\ &= E\{\|\mathbf{v}-\mathbf{y}\|^2\} + 2E\{(\mathbf{v}-\mathbf{y})^T(\mathbf{y}-\mathbf{y}')\} + E\{\|\mathbf{y}-\mathbf{y}'\|^2\} \end{aligned}$$

$(\mathbf{v}-\mathbf{y})$ is the quantization error and $(\mathbf{y}-\mathbf{y}')$ the truncation (mapping) error. These are now uncorrelated, so $E\{(\mathbf{v}-\mathbf{y})^T(\mathbf{y}-\mathbf{y}')\} = 0$. Thus

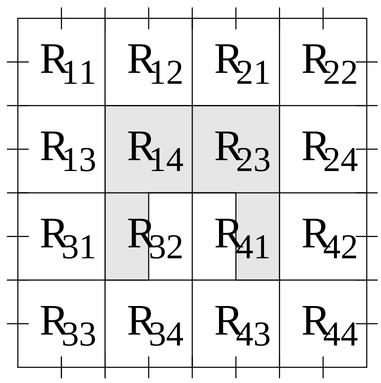
$$e_T^2 = E\{\|\mathbf{v}-\mathbf{y}\|^2\} + E\{\|\mathbf{y}-\mathbf{y}'\|^2\} = e_q^2 + e_m^2$$

3.

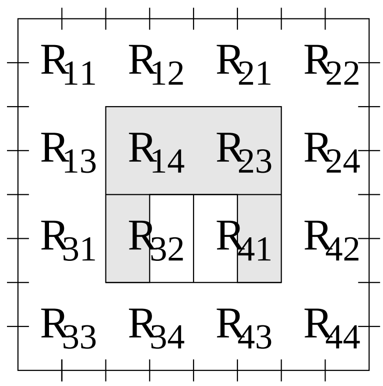
The given image is first divided into four regions:



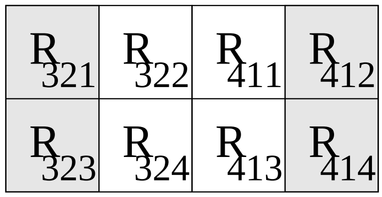
Each region is still non-homogeneous, so each region is further divided into four regions:



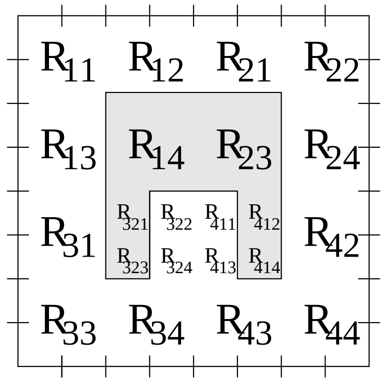
Now the border regions (R_{11}, R_{12}, ...) are all homogenous with the same intensity (i.e. white) and can therefore be merged into one region. The regions R_{14} and R_{23} are also homogenous with the same gray-scale value and are merged as well:



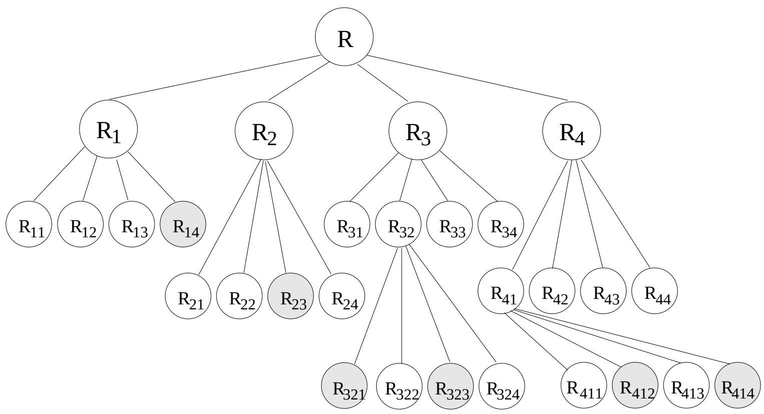
However, the regions R_{32} and R_{41} are not homogeneous and must be divided further. Here is a magnified image of the further division:



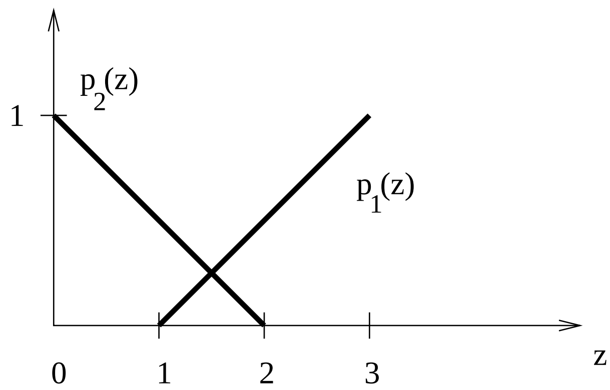
These regions are all homogeneous and can be merged into the two main regions from the previous step, and we are done:



Here is the requested quadtree:



4.



We obtain from the image the following distributions:

$$p_1(z) = \begin{cases} 0, & z < 1 \\ \frac{1}{2}z - \frac{1}{2}, & 1 \le z \le 3 \\ 0, & z > 3 \end{cases}$$

$$p_2(z) = \begin{cases} 0, & z < 0 \\ -\frac{1}{2}z + 1, & 0 \le z \le 2 \\ 0, & z > 2 \end{cases}$$

Because of the symmetry T=1.5. Theoretical proof follows:

$$\int_0^T p_1(z)dz$$

is the probability for object misclassification, and

$$\int_T^3 p_2(z)dz$$

is the probability for background misclassification.

So, the probability for misclassification is

$$E(T) = P_1 \int_0^T p_1(z)dz + P_2 \int_T^3 p_2(z)dz.$$

Next we minimize $E(T)$. We derive with respect to T and set the result to zero:

$$E'(T) = P_1 p_1(T) - P_2 p_2(T) = 0$$

Thus T must realize

$$P_1 p_1(T) = P_2 p_2(T)$$

Because $P_1 = P_2$,

$$\frac{1}{2}T - \frac{1}{2} = -\frac{1}{2}T + 1 \Leftrightarrow T = \frac{3}{2} = 1.5$$

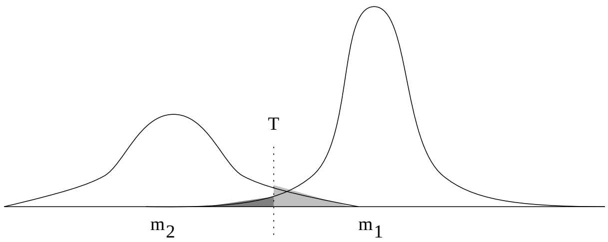
5.

The probability distribution for gray levels in class i is

$$p_i(z) = \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left[-\frac{(z-\mu_i)^2}{2\sigma_i^2}\right]$$

The probability $p(z)$ for the gray level z in the image is the joint probability of the gray levels of the bubbles and the background, $p(z) = P_1 p_1(z) + P_2 p_2(z)$. P_i is the *a priori* probability for the class i, i.e., the probability that a pixel belongs to class i without knowing its gray level value. So,

$$p(z) = P_1 \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left[-\frac{(z-\mu_1)^2}{2\sigma_1^2}\right] + P_2 \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left[-\frac{(z-\mu_2)^2}{2\sigma_2^2}\right]$$



If the threshold is T, the probability for the error is (see Figure)

$$E(T) = E_1(T) + E_2(T) = P_1 \int_{-\infty}^{T} p_1(z)dz + P_2 \int_{T}^{\infty} p_2(z)dz$$

The error function is minimized by setting its derivative to zero, $E'(T) = P_1 p_1(T) - P_2 p_2(T) = 0$, and we obtain

$$P_1 p_1(T) = P_2 p_2(T).$$

Because the exponentials may be tricky, we take the logarithm on both sides:

$$\ln P_1 p_1(T) - \ln P_2 p_2(T) = \ln \frac{P_1}{\sqrt{2\pi}\sigma_1} - \frac{(T-\mu_1)^2}{2\sigma_1^2} - \ln \frac{P_2}{\sqrt{2\pi}\sigma_2} + \frac{(T-\mu_2)^2}{2\sigma_2^2} = 0,$$

$$\Leftrightarrow -\sigma_2^2(T^2 - 2T\mu_1 + \mu_1^2) + \sigma_1^2(T^2 - 2T\mu_2 + \mu_2^2) + 2\sigma_1^2\sigma_2^2 \ln \frac{\sigma_2 P_1}{\sigma_1 P_2} = 0$$

$$\Leftrightarrow (\sigma_1^2 - \sigma_2^2)T^2 + 2(\mu_1\sigma_2^2 - \mu_2\sigma_1^2)T + \sigma_1^2\mu_2^2 - \sigma_2^2\mu_1^2 + 2\sigma_1^2\sigma_2^2 \ln \frac{\sigma_2 P_1}{\sigma_1 P_2} = 0.$$

When we now place the given values in the equation and observe also that $P_1/P_2 = 20\%/80\% = 0.25$, we obtain $-225T^2 - 167500T - 1.4506 \cdot 10^7 = 0$, and the final result is $\underline{T_1 \approx 100}$ and $\underline{T_2 \approx 644}$. The larger solution is due to the fact that the distribution of the bubbles is sharper (the variance is smaller) and on the right it will go (again!) under the distribution of the background.