T-61.5100 Digital image processing, Exercise 7/07

1.

A mean approximation pyramid is formed by forming 2×2 block averages. Since the starting image is of size 4×4, $J = 2$, and $f(x, y)$ is placed in level 2 of the mean approximation pyramid. The level 1 approximation is (by taking 2×2 block averages over $f(x, y)$ and subsampling):

$$\begin{bmatrix} 3.5 & 5.5 \\ 11.5 & 13.5 \end{bmatrix}$$

and the level 0 approximation is similarly [8.5]. The completed mean approximation pyramid is

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} 3.5 & 5.5 \\ 11.5 & 13.5 \end{bmatrix} \begin{bmatrix} 8.5 \end{bmatrix}.$$

Since no interpolation filtering is specified, pixel replication is used in the generation of the mean prediction residual pyramid levels. Level 0 of the prediction residual pyramid is the lowest resolution approximation, [8.5]. The level 2 prediction residual is obtained by upsampling the level 1 approximation and subtracting it from the level 2 (original image). Thus, we get

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} - \begin{bmatrix} 3.5 & 3.5 & 5.5 & 5.5 \\ 3.5 & 3.5 & 5.5 & 5.5 \\ 11.5 & 11.5 & 13.5 & 13.5 \\ 11.5 & 11.5 & 13.5 & 13.5 \end{bmatrix} = \begin{bmatrix} -2.5 & -1.5 & -2.5 & -1.5 \\ 1.5 & 2.5 & 1.5 & 2.5 \\ -2.5 & -1.5 & -2.5 & -1.5 \\ 1.5 & 2.5 & 1.5 & 2.5 \end{bmatrix}.$$

Similarly, the level 1 prediction residual is obtained by upsampling the level 0 approximation and subtracting it from the level 1 approximation to yield

$$\begin{bmatrix} 3.5 & 5.5 \\ 11.5 & 13.5 \end{bmatrix} - \begin{bmatrix} 8.5 & 8.5 \\ 8.5 & 8.5 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 3 & 5 \end{bmatrix}.$$

The mean prediction residual pyramid is therefore

$$\begin{bmatrix} -2.5 & -1.5 & -2.5 & -1.5 \\ 1.5 & 2.5 & 1.5 & 2.5 \\ -2.5 & -1.5 & -2.5 & -1.5 \\ 1.5 & 2.5 & 1.5 & 2.5 \end{bmatrix} \begin{bmatrix} -5 & -3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 8.5 \end{bmatrix}.$$

2.

The 2×2 Haar transformation matrix is

$$\mathbf{H}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then we get

$$\mathbf{T} = \mathbf{HFH}^{\mathrm{T}} = (\frac{1}{\sqrt{2}})^2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -3 & 0 \end{bmatrix}.$$

Next we compute the inverse Haar transform. First, the transpose of the 2×2 Haar transformation matrix is computed:

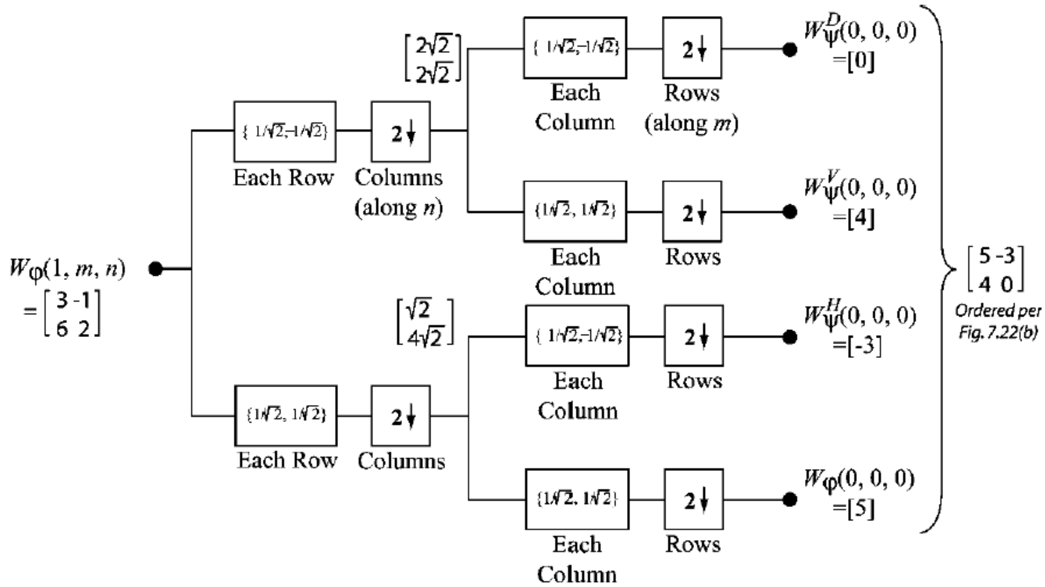
$$\mathbf{H}^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \mathbf{H}.$$

Then,

$$\mathbf{F} = \mathbf{H}^{\mathrm{T}}\mathbf{TH} = (\frac{1}{\sqrt{2}})^2 \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & 2 \end{bmatrix}.$$

3.

One pass through the FWT (fast wavelet transform) 2-d filter bank is all that is required:



4.

The set $\{\psi_{j,k}(x)\}$ of wavelets is defined as

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

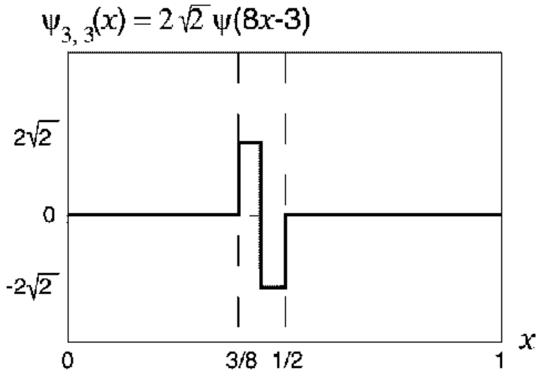
From this definition we obtain

$$\psi_{3,3}(x) = 2^{3/2}\psi(2^3 x - 3) = 2\sqrt{2}\psi(8x - 3).$$

Using the Haar wavelet function definition,

$$\psi(x) = \begin{cases} 1 & 0 \leq x < 0.5 \\ -1 & 0.5 \leq x < 1 \\ 0 & \text{elsewhere,} \end{cases}$$

we obtain the following plot.



To express $\psi_{3,3}(x)$ as a function of scaling functions, we notice that any wavelet function can be expressed as a sum of shifted, double-resolution scaling functions,

$$\psi(x) = \sum_n h_\psi(n)\sqrt{2}\varphi(2x - n).$$

Employing this result and the Haar wavelet vector defined in the textbook in Example 7.6 ($h_\psi(0) = 1/\sqrt{2}$ and $h_\psi(1) = -1/\sqrt{2}$), we get

$$\psi(8x-3) = \sum_n h_\psi(n)\sqrt{2}\varphi(2(8x-3)-n) = \frac{1}{\sqrt{2}}\sqrt{2}\varphi(16x-6)+\frac{-1}{\sqrt{2}}\sqrt{2}\varphi(16x-7) = \varphi(16x-6)-\varphi(16x-7).$$

Then, since $\psi_{3,3} = 2\sqrt{2}\psi(8x-3)$,

$$\psi_{3,3} = 2\sqrt{2}\psi(8x-3) = 2\sqrt{2}\varphi(16x-6) - 2\sqrt{2}\varphi(16x-7).$$

5.

The DWT transform pair is given as

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\varphi_{j_0,k}(x)$$
$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_x f(x)\psi_{j,k}(x),$$

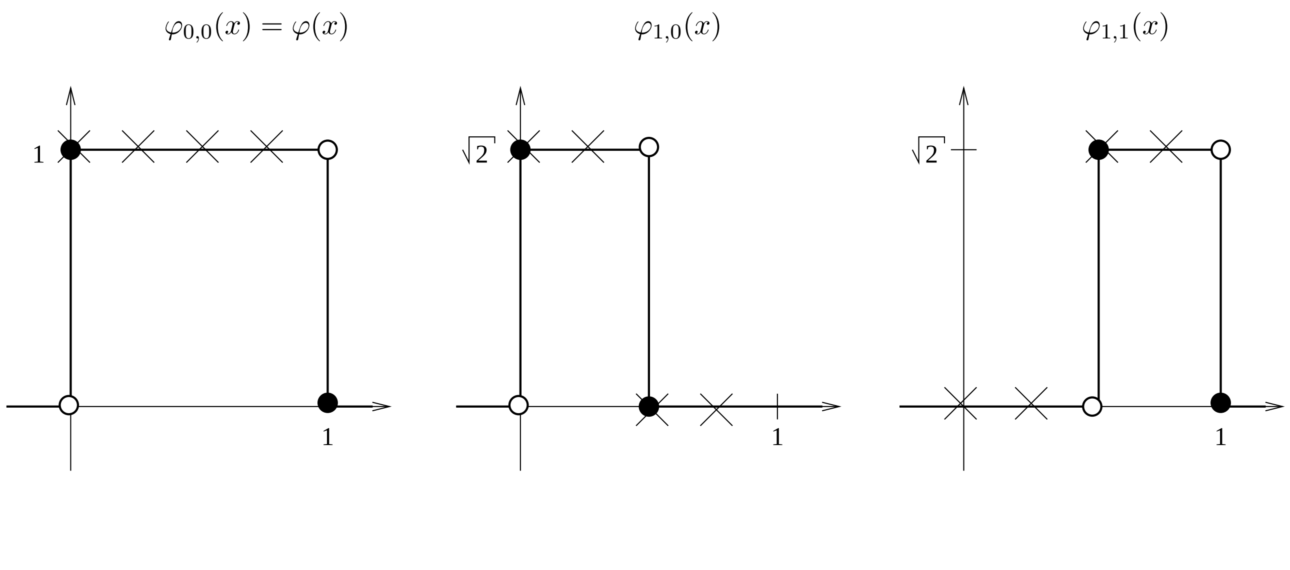
and the inverse transform as

$$f(x) = \frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\varphi_{j_0,k}(x) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\psi_{j,k}(x).$$

Now $M = 4$, $J = 2$, and $j_0 = 1$, so the summations in the above formulas are performed over $x = 0, 1, 2, 3$, $j = 1$, and $k = 0, 1$. Using Haar functions and assuming that they are distributed over the range of the input sequence, we get for the scaling functions

$$\begin{aligned}
\varphi_{1,0}(x) &= \sqrt{2}\varphi(2x) \\
\varphi_{1,1}(x) &= \sqrt{2}\varphi(2x-1), \\
\text{where} \quad \varphi(x) &= \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

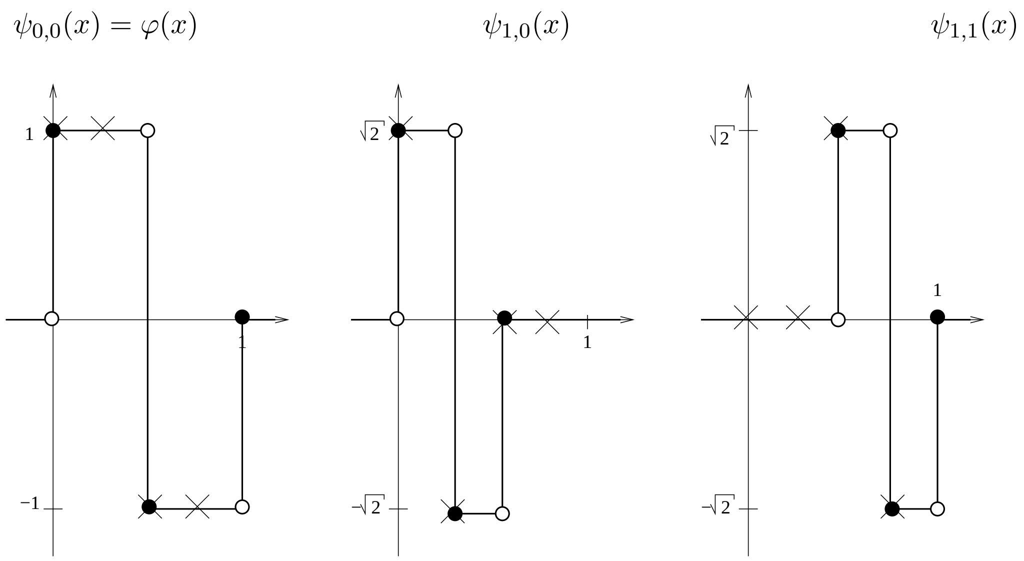
and below is shown the three functions with the four sampling points, i.e. from where we should read values for $x = 0, 1, 2, 3$:



Similarly for the wavelet functions

$$\begin{aligned}
\psi_{1,0}(x) &= \sqrt{2}\psi(2x) \\
\psi_{1,1}(x) &= \sqrt{2}\psi(2x-1), \\
\text{where} \quad \psi(x) &= \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

and the functions with the four sampling points:



Now we can calculate the transform,

$$\begin{aligned}
W_\varphi(1,0) &= \frac{1}{2}[f(0)\varphi_{1,0}(0) + f(1)\varphi_{1,0}(1) + f(2)\varphi_{1,0}(2) + f(3)\varphi_{1,0}(3)] \\
&= \frac{1}{2}[(1)(\sqrt{2}) + (4)(\sqrt{2}) + (-3)(0) + (0)(0)] = \frac{5\sqrt{2}}{2} \\
W_\varphi(1,1) &= \frac{1}{2}[f(0)\varphi_{1,1}(0) + f(1)\varphi_{1,1}(1) + f(2)\varphi_{1,1}(2) + f(3)\varphi_{1,1}(3)] \\
&= \frac{1}{2}[(1)(0) + (4)(0) + (-3)(\sqrt{2}) + (0)(\sqrt{2})] = \frac{-3\sqrt{2}}{2} \\
W_\psi(1,0) &= \frac{1}{2}[f(0)\psi_{1,0}(0) + f(1)\psi_{1,0}(1) + f(2)\psi_{1,0}(2) + f(3)\psi_{1,0}(3)] \\
&= \frac{1}{2}[(1)(\sqrt{2}) + (4)(-\sqrt{2}) + (-3)(0) + (0)(0)] = \frac{-3\sqrt{2}}{2} \\
W_\psi(1,1) &= \frac{1}{2}[f(0)\psi_{1,1}(0) + f(1)\psi_{1,1}(1) + f(2)\psi_{1,1}(2) + f(3)\psi_{1,1}(3)] \\
&= \frac{1}{2}[(1)(0) + (4)(0) + (-3)(\sqrt{2}) + (0)(-\sqrt{2})] = \frac{-3\sqrt{2}}{2}
\end{aligned}$$

so that the DWT is $5\sqrt{2}/2,\ \ -3\sqrt{2}/2,\ \ -3\sqrt{2}/2,\ \ -3\sqrt{2}/2.$