T-61.5100 Digital image processing, Exercise 5/07

1.

From Eq. (5.5-13),

$$g(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha, \beta)h(x - \alpha, y - \beta)d\alpha d\beta,$$

where

$$f(\alpha, \beta) = \delta(\alpha - a)$$

and

$$h(x - \alpha, y - \beta) = e^{-[(x-\alpha)^2+(y-\beta)^2]}.$$

Then we get

$$\begin{aligned} g(x, y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha - a)e^{-[(x-\alpha)^2+(y-\beta)^2]}d\alpha d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\alpha - a)e^{-[(x-\alpha)^2]}e^{-[(y-\beta)^2]}d\alpha d\beta \\ &= \int_{-\infty}^{\infty} \delta(\alpha - a)e^{-[(x-\alpha)^2]}d\alpha \int_{-\infty}^{\infty} e^{-[(y-\beta)^2]}d\beta \\ &= e^{-[(x-a)^2]} \int_{-\infty}^{\infty} e^{-[(y-\beta)^2]}d\beta, \end{aligned}$$

where we used the fact that the integral of the impulse is nonzero only when $\alpha = a$. Next, we note that

$$\int_{-\infty}^{\infty} e^{-[(y-\beta)^2]}d\beta = \int_{-\infty}^{\infty} e^{-[(\beta-y)^2]}d\beta$$

which is in the form of a constant times a Gaussian density with variance $\sigma^2 = 1/2$ or standard deviation $\sigma = 1/\sqrt{2}$. In other words,

$$e^{[-(y-\beta)^2]} = \sqrt{2\pi}\sqrt{1/2}\left[\frac{1}{\sqrt{2\pi}\sqrt{1/2}}e^{[-\frac{(\beta-y)^2}{2\,(1/2)}]}\right].$$

The integral from minus to plus infinity of the quantity inside the brackets is 1, so

$$g(x, y) = e^{[-(x-a)^2]}\sqrt{2\pi}\sqrt{1/2} = \sqrt{\pi}e^{[-(x-a)^2]}.$$

This the blurred version of the original image.

2.

Because the motion in the x- and y-directions are independent (motion is in the vertical (x) direction only at first, and then switching to motion only in the horizontal (y) direction) this problem can be solved in two steps. The first step is identical to the analysis that resulted in Eq. (5.6-10), which gives the blurring function due to vertical motion only:

$$H_1(u, v) = \frac{T_1}{\pi ua}\sin(\pi ua)e^{-j\pi ua},$$

where we are representing linear motion by the equation $x_0(t) = at/T_1$. The function $H_1(u, v)$ would give us a blurred image in the vertical direction. That blurred image is the image that would then start moving in the horizontal direction and to which horizontal blurring would be applied. This is nothing more than applying a second filter with transfer function

$$H_2(u, v) = \frac{T_2}{\pi vb}\sin(\pi vb)e^{-j\pi vb},$$

where we assumed the form $y_0(t) = bt/T_2$ for motion in the y-direction. Therefore, the overall blurring transfer function is given by the product of these two functions

$$H(u,v) = \frac{T_1 T_2}{(\pi ua)(\pi vb)} \sin(\pi ua) \sin(\pi vb) e^{-j\pi ua} e^{-j\pi vb},$$

and the overall blurred image is

$$g(x,y) = \mathcal{F}^{-1}[H(u,v)F(u,v)]$$

where $F(u,v)$ is the Fourier transform of the input image.

3.

The power spectrum of the restored image $\hat{f}$ is given by

$$S_{\hat{f}}(u,v) = |R(u,v)|^2 S_g(u,v).$$

The restoration filter should force the power spectrum of the restored image to equal the power spectrum of the original image:

$$S_{\hat{f}}(u,v) = S_f(u,v).$$

We start with the above equations:

$$\begin{aligned} S_f(u,v) &= |R(u,v)|^2 S_g(u,v) = |R(u,v)|^2 |G(u,v)|^2 \\ &= |R(u,v)|^2 \left[|H(u,v)|^2 |F(u,v)|^2 + |N(u,v)|^2 + \right. \\ &\quad \left. H(u,v) N^*(u,v) F(u,v) + H^*(u,v) F^*(u,v) N(u,v) \right] \\ &= |R(u,v)|^2 \left[|H(u,v)|^2 S_f(u,v) + S_\eta(u,v) \right], \end{aligned}$$

where the cross-terms vanish because the image and the noise are uncorrelated. $S_\eta(u,v)$ is the power spectrum of the noise. Then we solve for $|R(u,v)|$:

$$|R(u,v)| = \sqrt{\frac{1}{|H(u,v)|^2 + \frac{S_\eta(u,v)}{S_f(u,v)}}}$$

4.

Inverse filter:

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

So the magnitude response of the inverse filter is

$$|H_{\text{INV}}(u,v)| = \frac{1}{|H(u,v)|}.$$

Power spectrum equalization filter:

$$|H_{\text{PSE}}(u,v)| = \left[\frac{1}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)} \right]^{1/2}$$

Wiener-filter:

$$\hat{F}(u,v) = \left[\frac{H^\star(u,v)}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)} \right] G(u,v),$$

where

$$|H_{\text{WIENER}}(u,v)| = \frac{|H(u,v)|}{|H(u,v)|^2 + S_\eta(u,v)/S_f(u,v)}$$

Let substitute the given values into these equations:

$\lvert H(u,v)\rvert$	$S_f(u,v)$	$S_\eta(u,v)$	$\lvert H_{\text{INV}}(u,v)\rvert$	$\lvert H_{\text{PSE}}(u,v)\rvert$	$\lvert H_{\text{WIENER}}(u,v)\rvert$
0	0	N	∞	0	0
0	S	0	∞	∞	$0/0$, usually 0
0	S	N	∞	$\sqrt{S/N}$	0
H	0	N	$1/H$	0	0
H	S	0	$1/H$	$1/H$	$1/H$
1.0	3000	0.01	1.0	≈ 1.0	≈ 1.0
0.7	0.7	0.01	1.43	≈ 1.41	≈ 1.38
0.01	0.005	0.01	100.0	≈ 0.71	≈ 0.005

5.

a) The 3×3-sized local mean mask is (scaling is omitted for simplicity)

1	1	1
1	1	1
1	1	1

The part of an image that falls under the mask is given as

a	b	c
d	e	f
g	h	i

The mask response in position e is $e^* = 1\cdot a + 1\cdot b + 1\cdot c + 1\cdot d + 1\cdot e + 1\cdot f + 1\cdot g + 1\cdot h + 1\cdot i$ $= a+b+c+d+e+f+g+h+i$. If we are using the mask $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}$, the responses in positions b, e, and h are $b' = a+b+c$, $e' = d+e+f$, and $h' = g+h+i$. When we then apply the mask $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array}^T$, the response in position e is $e'' = b'+e'+h' = (a+b+c)+(d+e+f)+(g+h+i) = e^*$.

With a 3×3 mask we have 8 additions for each mask position. With a 1×3 mask we have 2 additions for each position. Thus using 1×3 and 3×1 masks takes a total of 4 additions for each position which is half the number of additions needed with a 3×3 mask.

b) In a general case we have N^2-1 additions and N^2 multiplications with a N×N mask. Using the separate masks takes $2(N-1)$ additions and $2N$ multiplications.

c) The 3×3 Sobel gradient masks are

$$G_x = \begin{array}{|c|c|c|}\hline -1 & -2 & -1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 2 & 1 \\ \hline \end{array} \qquad\qquad G_y = \begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -2 & 0 & 2 \\ \hline -1 & 0 & 1 \\ \hline \end{array}$$

G_x measures horizontal edges and G_y vertical edges.

Let us use the G_x mask. The response in position e is $e^* = (g+2h+i)-(a+2b+c)$. With a one-dimensional difference mask $\begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline \end{array}^T$ the responses in positions d, e, and f are $d' = g-a$, $e' = h-b$, and $f' = i-c$. When we then apply the mask $\begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline \end{array}$ the response in position e is $e'' = (g-a)+2(h-b)+(i-c) = (g+2h+i)-(a+2b+c) = e^*$. G_y mask can be used in the same way.

d) The 3×3 discrete Laplace-operator is

0	-1	0
-1	4	-1
0	-1	0

In order to separate the 3×3 discrete Laplace-operator, we must find two 3×1 vectors $\begin{bmatrix} a & b & c \end{bmatrix}^T$ and $\begin{bmatrix} d & e & f \end{bmatrix}^T$ whose outer product

ad	bd	cd
ae	be	ce
af	bf	cf

were the Laplace mask. For example, $ad = 0$. If we choose $a = 0$ then also $ae = 0$ which is not valid. If $d = 0$ then $bd = 0$ which is also not valid. So, we cannot separate the mask into two one-dimensional vectors.