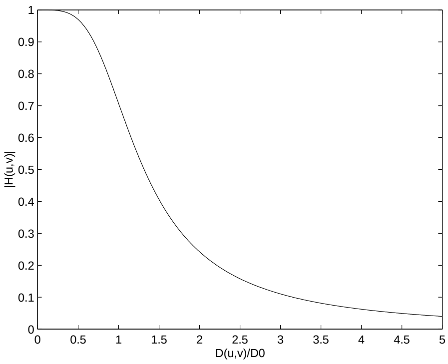
T-61.5100 Digital image processing, Exercise 4/07

1.

The frequency response of the Butterworth filter is flat in both the passband and the stopband. The transition band is usually quite wide, and the filter order must be higher than in other common filters if we wish to fulfill the required specifications.
The Butterworth lowpass filter is given as

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}},$$

where $D(u,v) = \sqrt{u^2 + v^2}$. Its frequency response looks like that:



<u>The answer:</u> Never. The Butterworth filter approaches the ideal filter when $n \to \infty$, because

$$\lim_{n\to\infty} |H(u,v)| = \begin{cases} 1, & \text{when } D(u,v) < D_0 \\ 0, & \text{when } D(u,v) > D_0 \end{cases}$$

In fact $H(u,v) = 1/2$ always if $D(u,v) = D_0$, so the filter does not *exactly* approach the ideal filter.

2.

Let $h(x,y)$ be an $m \times m$-sized spatial mask. Let $\mathbf{h}$ be the mask ordered as an m^2-sized vector, e.g.,

$$\mathbf{h}_{(x-1)m+y} = h(x,y).$$

The Fourier transform is linear. Let the complex transformation matrix be $\mathbf{F}$, and the transformed mask (N^2-sized complex vector) be $\hat{\mathbf{H}}$:

$$\hat{\mathbf{H}} = \mathbf{F}\mathbf{h}.$$

Now we want $\hat{\mathbf{H}}$ to be as close as possible to the Butterworth filter $\mathbf{H}$ (also N^2-sized complex vector). In effect we want to minimize the error function,

$$\mathcal{E}(\mathbf{h}) = \|\hat{\mathbf{H}} - \mathbf{H}\|^2 = \|\mathbf{F}\mathbf{h} - \mathbf{H}\|^2.$$

The minimum can be found where the partial derivative vanishes:

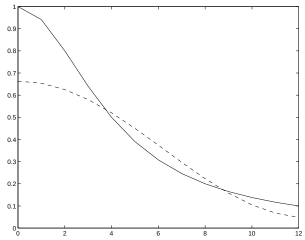
$$\frac{\partial \mathcal{E}(\mathbf{h})}{\partial \mathbf{h}} = 2\mathbf{F}^T(\mathbf{F}\mathbf{h} - \mathbf{H}) = 0.$$

This is solved when $\mathbf{F}\mathbf{h} - \mathbf{H} = 0$, i.e.

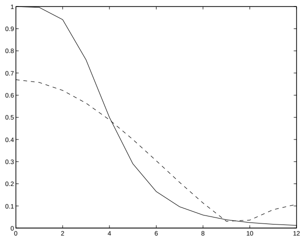
$$\mathbf{h} = \mathbf{F}^\dagger \mathbf{H},$$

where $\mathbf{F}^\dagger = (\mathbf{F}^T\mathbf{F})^{-1}\mathbf{F}^T$ is the pseudo-inverse of $\mathbf{F}$.
The Matlab-code that generates the spatial mask for the Butterworth filter can be found in `http://www.cis.hut.fi/Opinnot/T-61.5100/spatial.m`

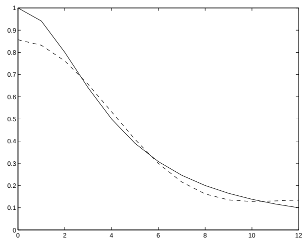
Results:



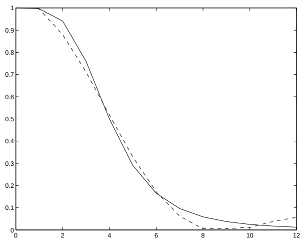
0.0404	0.0736	0.0404
0.0736	0.2075	0.0736
0.0404	0.0736	0.0404



0.0576	0.0800	0.0576
0.0800	0.1196	0.0800
0.0576	0.0800	0.0576



0.0067	0.0130	0.0156	0.0130	0.0067
0.0130	0.0404	0.0736	0.0404	0.0130
0.0156	0.0736	0.2075	0.0736	0.0156
0.0130	0.0404	0.0736	0.0404	0.0130
0.0067	0.0130	0.0156	0.0130	0.0067



0.0107	0.0244	0.0318	0.0244	0.0107
0.0244	0.0576	0.0800	0.0576	0.0244
0.0318	0.0800	0.1196	0.0800	0.0318
0.0244	0.0576	0.0800	0.0576	0.0244
0.0107	0.0244	0.0318	0.0244	0.0107

Solid line: the frequency response of the Butterworth filter.
Dotted line: the frequency response of the spatial mask.
Left column: $n = 1$. Right column: $n = 2$.
Notice that in both cases the 3×3 mask is included in the 5×5 mask. This is a general property that comes from the orthogonality. If the mask size is increased, the obtained frequency response approaches the frequency response of the Butterworth filter.

3.

a) The filtered image is given by

$$g(x, y) = h(x, y) * f(x, y),$$

where $h(x, y)$ is the spatial filter (inverse Fourier transform of the frequency-domain filter) and f is the input image. Histogram processing this result yields

$$g'(x, y) = T[g(x, y)] = T[h(x, y) * f(x, y)],$$

where T denotes the histogram equalization transformation. If we histogram-equalize first, then

$$g(x, y) = T[f(x, y)]$$

and

$$g'(x, y) = h(x, y) * T[f(x, y)].$$

In general, T is a nonlinear function determined by the nature of the pixels in the image from which it is computed. Thus, in general $T[h(x, y) * f(x, y)] \neq h(x, y) * T[f(x, y)]$ and the order does matter.

b) As indicated in Section 4.4, highpass filtering severely diminishes the contrast of an image. Although high-frequency emphasis helps some, the improvement is usually not dramatic. Thus, if an image is histogram equalized first, the gain in contrast improvement will essentially be lost in the filtering process. Therefore, the procedure in general is *to filter first and histogram-equalize the image after that.*
A small example: Let us assume that the following image has two areas with even intensities, i_1 and i_2.

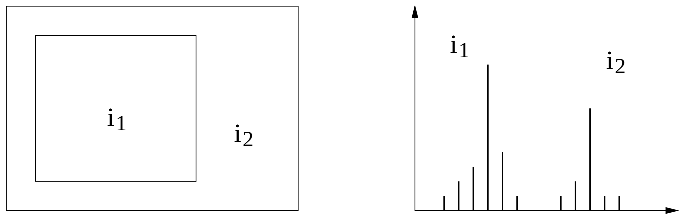
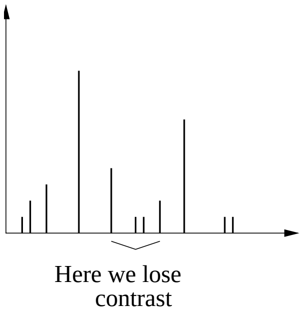


Image   Histogram of the imag

These areas are shown in the histogram as high peaks and in the frequency spectrum as low frequencies. Changes in intensities (high frequencies) in these areas and in their edges are typically shown as lower peaks around the high peaks. If we now emphasize the high frequencies, we in fact emphasize those lower peaks, and thus we can make more use of these high frequencies in histogram equalization. If we do the histogram equalization first, it is more based on the high peaks in the histogram. Then we might lose some contrast in the final image:



4.

Let us assume that there is only a single star that is modeled as an impulse $\delta(x - x_0, y - y_0)$ where (x_0, y_0) are the coordinates of the star. (A discrete impulse is one at the origin, zero elsewhere.) K is the illumination and ϵ the reflectance of the sky. Then the image model is

$$f(x, y) = i(x, y)r(x, y) = K(\delta(x - x_0, y - y_0) + \epsilon)$$

Homomorphic filtering:

$$f(x, y) \rightarrow \boxed{\ln} \rightarrow \boxed{\mathcal{F}} \rightarrow \boxed{H(u, v)} \rightarrow \boxed{\mathcal{F}^{-1}} \rightarrow \boxed{\exp} \rightarrow g(x, y)$$

So, first we take the logarithm from the image:

$$z(x, y) = \ln f(x, y) = \ln K + \ln(\delta(x - x_0, y - y_0) + \epsilon) = \ln K + \ln \epsilon + \delta(x - x_0, y - y_0)(\ln(1 + \epsilon) - \ln \epsilon).$$

Then we Fourier transform:

$$\mathcal{F}\{z(x, y)\} = \mathcal{F}\{\ln K\epsilon\} + \mathcal{F}\{\delta(x - x_0, y - y_0)(\ln(1 + \epsilon) - \ln \epsilon)\}$$

$$= C_1\delta(u, v) + C_2 e^{-j2\pi(\frac{ux_0}{M} + \frac{vy_0}{N})},$$

where $C_1 = \ln K\epsilon$ ja $C_2 = \ln(1 + \epsilon) - \ln \epsilon$.

From this result, it is evident that the contribution of illumination is an impulse at the origin of the frequency plane. It can be canceled by high-pass filtering the image (e.g. using a notch filter). Extension of this development to multiple impulses (stars) is straightforward. The filter will be the same.

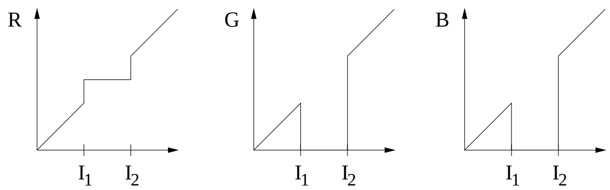
5.

a) Perform a median filtering operation – an isolated bright dot won't greatly affect the median value.

b) Sharpness can be enhanced by high-frequency emphasis. Then areas with high frequencies (edges) will be emphasized, but areas with low frequencies (even areas) are canceled.

c) Contrast can be enhanced by histogram equalization.

d) Compute the average gray level, K_0. Add the quantity $(K - K_0)$ to all pixels.
Coloring is achieved with the following color transformation:



So only the red color remains in $I_1 \ldots I_2$. Here R, G, and B are the color components of an RGB color monitor.