

## T-61.5100 Digital image processing, Exercise 3/07

1.

These will be needed:

$$\sum_{x=0}^{N-1} q^x = \frac{1 - q^N}{1 - q}, \quad e^{jx} = \cos x + j \sin x$$

First  $r = u + kN$ :

$$\sum_{x=0}^{N-1} e^{j2\pi rx/N} e^{-j2\pi ux/N} = \sum_{x=0}^{N-1} e^{j2\pi(r-u)x/N} = \sum_{x=0}^{N-1} e^{j2\pi kx} = \sum_{x=0}^{N-1} \underbrace{\cos 2\pi kx}_1 + j \underbrace{\sin 2\pi kx}_0 = N$$

Next  $r \neq u + kN$ :

$$\sum_{x=0}^{N-1} e^{j2\pi(r-u)x/N} = \sum_{x=0}^{N-1} \left( e^{j2\pi(r-u)/N} \right)^x$$

This is a geometric sum. Thus

$$= \frac{1 - \left( e^{j2\pi(r-u)/N} \right)^N}{1 - \left( e^{j2\pi(r-u)/N} \right)} = \frac{1 - e^{j2\pi(r-u)}}{1 - e^{j2\pi(r-u)/N}} = \frac{1 - \overbrace{\cos 2\pi(r-u)}^1 - j \overbrace{\sin 2\pi(r-u)}^0}{\underbrace{1 - e^{j2\pi(r-u)/N}}_{\neq 0 \text{ always, provided that } r-u \neq kN, k \in \mathbb{Z}}} = 0$$

2.

The discrete Fourier-transform and the inverse transform:

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-j2\pi ux/N}, \quad f(x) = \sum_{u=0}^{N-1} F(u) e^{j2\pi ux/N}$$

The sequence is now  $f(0) = 2, f(1) = 3, f(2) = 4, f(3) = 4$ . The Fourier-transform is done using 4 points  $\implies N = 4$ :

$$F(0) = \frac{1}{4} \sum_{x=0}^3 f(x) e^0 = \frac{1}{4} [f(0) + f(1) + f(2) + f(3)] = \frac{1}{4} (2 + 3 + 4 + 4) = 3.25$$

$$\begin{aligned} F(1) &= \frac{1}{4} \sum_{x=0}^3 f(x) e^{-j2\pi x/4} = \frac{1}{4} (2e^0 + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}) \\ &= \frac{1}{4} [2 + 3(\cos(\pi/2) - j\sin(\pi/2)) + 4(\cos(\pi) - j\sin(\pi)) + 4(\cos(3\pi/2) - j\sin(3\pi/2))] \\ &= \frac{1}{4} [2 + 3(-j) + 4(-1) + 4(j)] = \frac{1}{4} (-2 + j) \end{aligned}$$

$$F(2) = \frac{1}{4} \sum_{x=0}^3 f(x) e^{-j2\pi 2x/4} = -\frac{1}{4} (1 + j0) = -\frac{1}{4}$$

$$F(3) = \frac{1}{4} \sum_{x=0}^3 f(x) e^{-j2\pi 3x/4} = -\frac{1}{4} (2 + j)$$

Then we do the inverse transform:

$$\begin{aligned}
 f(0) &= \sum_{u=0}^3 F(u)e^0 = \frac{13}{4} + \frac{1}{4}(-2+j) - \frac{1}{4} - \frac{1}{4}(2+j) = 2 \\
 f(1) &= \sum_{u=0}^3 F(u)e^{j2\pi u/4} = \frac{13}{4} + \frac{1}{4}(-2+j)e^{j\pi/2} - \frac{1}{4}e^{j\pi} - \frac{1}{4}(2+j)e^{j3\pi/2} = 3 \\
 f(2) &= \sum_{u=0}^3 F(u)e^{j2\pi 2u/4} = \frac{13}{4} + \frac{1}{4}(-2+j)(-1) - \frac{1}{4} - \frac{1}{4}(2+j)(-1) = 4 \\
 f(3) &= \sum_{u=0}^3 F(u)e^{j2\pi 3u/4} = \frac{13}{4} + \frac{1}{4}(-2+j)(-j) - \frac{1}{4}(-1) - \frac{1}{4}(2+j)j = 4
 \end{aligned}$$

The result is thus the original sequence.

### 3.

Convolution:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha)g(x - \alpha) d\alpha$$

The Fourier transform of a delay:

$$\mathcal{F}\{g(x - \alpha)\} = G(u) \cdot e^{-j2\pi u\alpha}$$

By direct substitution into the definition, we get

$$\begin{aligned}
 \mathcal{F}\{f(x) * g(x)\} &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\alpha)g(x - \alpha) d\alpha \right] e^{-j2\pi ux} dx \\
 &= \int_{-\infty}^{\infty} f(\alpha) \left[ \int_{-\infty}^{\infty} g(x - \alpha)e^{-j2\pi ux} dx \right] d\alpha = G(u) \int_{-\infty}^{\infty} f(\alpha)e^{-j2\pi u\alpha} d\alpha = G(u)F(u)
 \end{aligned}$$

This is a very practical result, since it is often easier to calculate separately the transforms of  $f$  and  $g$  by FFT, multiply them and calculate the inverse transform than to calculate the convolution straight from the definition.

### 4.

We start with only one variable and show first that, if

$$H(u) = e^{-u^2/2\sigma^2}$$

then

$$h(x) = \int_{-\infty}^{\infty} e^{-u^2/2\sigma^2} e^{j2\pi ux} du = \sqrt{2\pi}\sigma e^{-2\pi^2 x^2 \sigma^2}.$$

We can express the integral in the preceding equation as

$$h(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u^2 - j4\pi\sigma^2 ux]} du.$$

We now make use of the identity

$$e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} e^{\frac{(2\pi)^2 x^2 \sigma^2}{2}} = 1.$$

Inserting this identity in the preceding integral yields

$$h(x) = e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [u^2 - j4\pi\sigma^2 ux - (2\pi)^2 \sigma^4 x^2]} du = e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2} [u - j2\pi\sigma^2 x]^2} du.$$

Next we make the change of variable  $r = u - j2\pi\sigma^2 x$ . Then,  $dr = du$  and the above integral becomes

$$h(x) = e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr.$$

Finally, we multiply and divide the right side of this equation by  $\sqrt{2\pi}\sigma$ :

$$h(x) = \sqrt{2\pi}\sigma e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \left[ \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr \right].$$

The expression inside the brackets is recognized as a Gaussian probability density function, whose integral from  $-\infty$  to  $\infty$  is 1. Therefore,

$$h(x) = \sqrt{2\pi}\sigma e^{-2\pi^2 \sigma^2 x^2}.$$

With this result as background, we now move into two-dimensional case. By substituting directly into definition of the inverse Fourier transform we have:

$$h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A e^{-(u^2+v^2)/2\sigma^2} e^{j2\pi(ux+vy)} du dv = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} A e^{(-\frac{u^2}{2\sigma^2} + j2\pi ux)} du \right] e^{(-\frac{v^2}{2\sigma^2} + j2\pi vy)} dv$$

By using the obtained result for the one-dimensional case, we now have the final result:

$$h(x, y) = (A\sqrt{2\pi}\sigma e^{-2\pi^2 \sigma^2 x^2}) (\sqrt{2\pi}\sigma e^{-2\pi^2 \sigma^2 y^2}) = A2\pi\sigma^2 e^{-2\pi^2 \sigma^2 (x^2+y^2)}.$$

## 5.

The spatial average is

$$g(x, y) = \frac{1}{4} [f(x, y+1) + f(x+1, y) + f(x, y-1) + f(x-1, y)].$$

We need the following property of the Fourier transform:

$$\mathcal{F}\{f(x-x_0, y-y_0)\} = \exp[-j2\pi(ux_0 + vy_0)/N] F(u, v)$$

a)

$$G(u, v) = \frac{1}{4} \left[ e^{-j2\pi v/N} + e^{-j2\pi u/N} + e^{j2\pi v/N} + e^{j2\pi u/N} \right] F(u, v) = H(u, v) F(u, v).$$

Because

$$e^{-jx} = \cos x - j \sin x \quad \text{and} \quad e^{jx} = \cos x + j \sin x$$

we get

$$H(u, v) = \frac{1}{2} \left[ \cos \frac{2\pi u}{N} + \cos \frac{2\pi v}{N} \right]$$

that is the filter transfer function in the frequency domain. ( $H(u, v)$  is real because the filter is symmetric!)

b)  $|H(0,0)| = 1$  and  $|H|$  decreases as a function of distance from the origin. This is the characteristic of a lowpass filter. In fact this happens monotonically in the directions of  $u$  and  $v$  axis only; in the diagonal directions also the higher frequencies are emphasized:

$$\left| H\left(\pm\frac{N}{2}, \pm\frac{N}{2}\right) \right| = |-1| = 1$$

Look at the images!

c) The phase response can be calculated from  $H(u, v)$ . Because  $\text{Im}\{H(u, v)\} = 0$ , then the phase response is 0 when  $\text{Re}\{H(u, v)\} > 0$ , and  $\pi$  where  $\text{Re}\{H(u, v)\} < 0$ :

$$\arg H(u, v) = \begin{cases} 0, & \text{if } (|u| + |v|)/N \leq 0.5 \\ \pi, & \text{if } (|u| + |v|)/N > 0.5 \end{cases}$$

Look at the images!

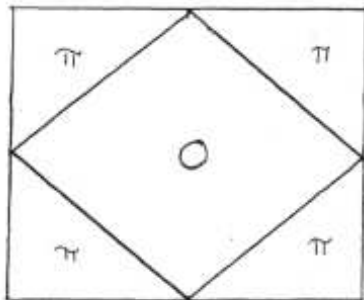
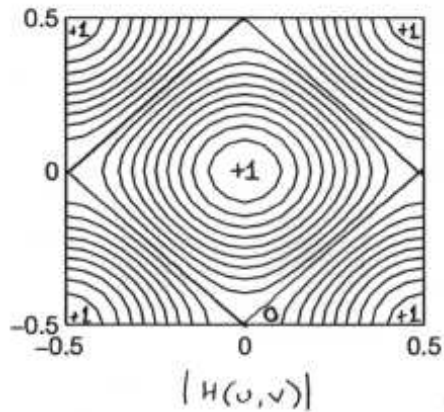
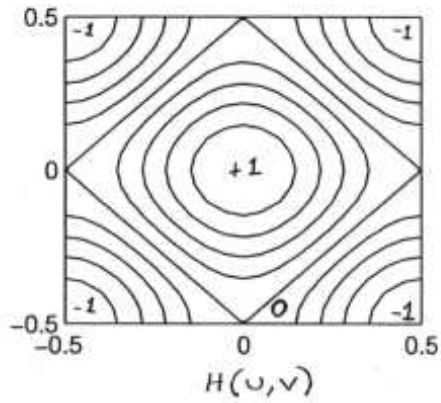
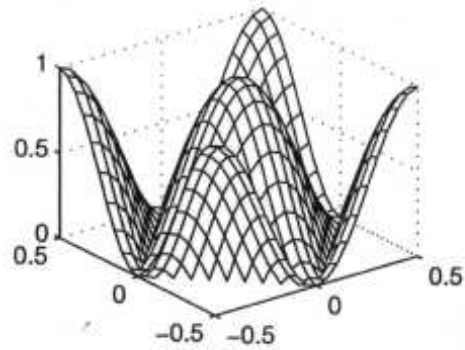
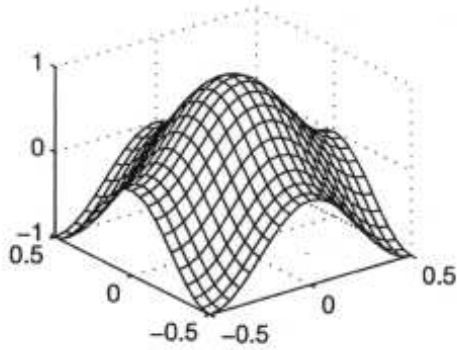
So  $H$  retains (but turns) “the chessboard” ( $u = v = N/2$ ) but averages the horizontal and vertical lines in images.

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 For information on all of the MathWorks products, type tour.

```

>> u=-0.5:.05:0.5;
>> v=u;
>> [X,Y]=meshdom(u,v);
>> Z=0.5*(cos(2*pi.*X)+cos(2*pi.*Y));
>> subplot(3,2,1), mesh(u,v,Z);
>> subplot(3,2,2), mesh(u,v,abs(Z));
>> subplot(3,2,3), contour(u,v,Z);
>> subplot(3,2,4), contour(u,v,abs(Z));
>> subplot(3,2,5), mesh(u,v,atan2(imag(Z),real(Z)));
>> print -dps
  
```



$\text{arg } H(u,v)$