T-61.5100 Digital image processing, Exercise 3/07

1.

These will be needed:

$$\sum_{x=0}^{N-1} q^x = \frac{1-q^N}{1-q}, \quad e^{jx} = \cos x + j \sin x$$

First $r = u + kN$:

$$\sum_{x=0}^{N-1} e^{j2\pi rx/N} e^{-j2\pi ux/N} = \sum_{x=0}^{N-1} e^{j2\pi(r-u)x/N} = \sum_{x=0}^{N-1} e^{j2\pi kx} = \sum_{x=0}^{N-1} \underbrace{\cos 2\pi kx}_{1} + j \underbrace{\sin 2\pi kx}_{0} = N$$

Next $r \neq u + kN$:

$$\sum_{x=0}^{N-1} e^{j2\pi(r-u)x/N} = \sum_{x=0}^{N-1} \left(e^{j2\pi(r-u)/N}\right)^x$$

This is a geometric sum. Thus

$$= \frac{1-\left(e^{j2\pi(r-u)/N}\right)^N}{1-\left(e^{j2\pi(r-u)/N}\right)} = \frac{1-e^{j2\pi(r-u)}}{1-e^{j2\pi(r-u)/N}} = \frac{1-\overbrace{\cos 2\pi(r-u)}^{1} - j\overbrace{\sin 2\pi(r-u)}^{0}}{\underbrace{1-e^{j2\pi(r-u)/N}}_{\neq 0 \text{ always, provided that } r-u\neq kN,\ k\in Z}} = 0$$

2.

The discrete Fourier-transform and the inverse transform:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)e^{-j2\pi ux/N}, \quad f(x) = \sum_{u=0}^{N-1} F(u)e^{j2\pi ux/N}$$

The sequence is now $f(0) = 2, f(1) = 3, f(2) = 4, f(3) = 4$. The Fourier-transform is done using 4 points $\Longrightarrow$ $N = 4$:

$$\begin{aligned}
F(0) &= \frac{1}{4}\sum_{x=0}^{3} f(x)e^0 = \frac{1}{4}[f(0)+f(1)+f(2)+f(3)] = \frac{1}{4}(2+3+4+4) = 3.25 \\
F(1) &= \frac{1}{4}\sum_{x=0}^{3} f(x)e^{-j2\pi x/4} = \frac{1}{4}(2e^0 + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}) \\
&= \frac{1}{4}[2 + 3(\cos(\pi/2) - j\sin(\pi/2)) + 4(\cos(\pi) - j\sin(\pi)) + 4(\cos(3\pi/2) - j\sin(3\pi/2))] \\
&= \frac{1}{4}[2 + 3(-j) + 4(-1) + 4(j)] = \frac{1}{4}(-2+j) \\
F(2) &= \frac{1}{4}\sum_{x=0}^{3} f(x)e^{-j2\pi 2x/4} = -\frac{1}{4}(1+j0) = -\frac{1}{4} \\
F(3) &= \frac{1}{4}\sum_{x=0}^{3} f(x)e^{-j2\pi 3x/4} = -\frac{1}{4}(2+j)
\end{aligned}$$

Then we do the inverse transform:

$$
\begin{aligned}
f(0) &= \sum_{u=0}^{3} F(u)e^{0} = \frac{13}{4} + \frac{1}{4}(-2+j) - \frac{1}{4} - \frac{1}{4}(2+j) = 2 \\
f(1) &= \sum_{u=0}^{3} F(u)e^{j2\pi u/4} = \frac{13}{4} + \frac{1}{4}(-2+j)e^{j\pi/2} - \frac{1}{4}e^{j\pi} - \frac{1}{4}(2+j)e^{j3\pi/2} = 3 \\
f(2) &= \sum_{u=0}^{3} F(u)e^{j2\pi 2u/4} = \frac{13}{4} + \frac{1}{4}(-2+j)(-1) - \frac{1}{4} - \frac{1}{4}(2+j)(-1) = 4 \\
f(3) &= \sum_{u=0}^{3} F(u)e^{j2\pi 3u/4} = \frac{13}{4} + \frac{1}{4}(-2+j)(-j) - \frac{1}{4}(-1) - \frac{1}{4}(2+j)j = 4
\end{aligned}
$$

The result is thus the original sequence.

3.

Convolution:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)\,d\alpha$$

The Fourier transform of a delay:

$$\mathcal{F}\{g(x-\alpha)\} = G(u)\cdot e^{-j2\pi u\alpha}$$

By direct substitution into the definition, we get

$$\mathcal{F}\{f(x) * g(x)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\alpha)g(x-\alpha)\,d\alpha\right]e^{-j2\pi ux}\,dx$$

$$= \int_{-\infty}^{\infty} f(\alpha)\left[\int_{-\infty}^{\infty} g(x-\alpha)e^{-j2\pi ux}\,dx\right]d\alpha = G(u)\int_{-\infty}^{\infty} f(\alpha)e^{-j2\pi u\alpha}\,d\alpha = G(u)F(u)$$

This is a very practical result, since it is often easier to calculate separately the transforms of f and g by FFT, multiply them and calculate the inverse transform than to calculate the convolution straight from the definition.

4.

We start with only one variable and show first that, if

$$H(u) = e^{-u^2/2\sigma^2}$$

then

$$h(x) = \int_{-\infty}^{\infty} e^{-u^2/2\sigma^2}e^{j2\pi ux}du = \sqrt{2\pi}\sigma e^{-2\pi^2x^2\sigma^2}.$$

We can express the integral in the preceding equation as

$$h(x) = \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u^2 - j4\pi\sigma^2 ux]}du.$$

We now make use of the identity

$$e^{-\frac{(2\pi)^2x^2\sigma^2}{2}}e^{\frac{(2\pi)^2x^2\sigma^2}{2}} = 1.$$

Inserting this identity in the preceding integral yields

$$h(x) = e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u^2 - j4\pi\sigma^2 ux - (2\pi)^2\sigma^4 x^2]} du = e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}[u - j2\pi\sigma^2 x]^2} du.$$

Next we make the change of variable $r = u - j2\pi\sigma^2 x$. Then, $dr = du$ and the above integral becomes

$$h(x) = e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr.$$

Finally, we multiply and divide the right side of this equation by $\sqrt{2\pi}\sigma$:

$$h(x) = \sqrt{2\pi}\sigma e^{-\frac{(2\pi)^2 x^2 \sigma^2}{2}} [\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{r^2}{2\sigma^2}} dr].$$

The expression inside the brackets is recognized as a Gaussian probability density function, whose integral from $-\infty$ to ∞ is 1. Therefore,

$$h(x) = \sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 x^2}.$$

With this result as background, we now move into two-dimensional case. By substituting directly into definition of the inverse Fourier transform we have:

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A e^{-(u^2+v^2)/2\sigma^2} e^{j2\pi(ux+vy)} du dv = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} A e^{(-\frac{u^2}{2\sigma^2} + j2\pi ux)} du \right] e^{(-\frac{v^2}{2\sigma^2} + j2\pi vy)} dv$$

By using the obtained result for the one-dimensional case, we now have the final result:

$$h(x,y) = (A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 x^2})(\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2 y^2}) = A2\pi\sigma^2 e^{-2\pi^2\sigma^2(x^2+y^2)}.$$

5.

The spatial average is

$$g(x,y) = \frac{1}{4}\left[f(x,y+1) + f(x+1,y) + f(x,y-1) + f(x-1,y)\right].$$

We need the following property of the Fourier transform:

$$\mathcal{F}\left\{f(x-x_0, y-y_0)\right\} = \exp\left[-j2\pi(ux_0 + vy_0)/N\right] F(u,v)$$

a)

$$G(u,v) = \frac{1}{4}\left[e^{-j2\pi v/N} + e^{-j2\pi u/N} + e^{j2\pi v/N} + e^{j2\pi u/N}\right] F(u,v) = H(u,v)F(u,v).$$

Because

$$e^{-jx} = \cos x - j\sin x \quad \text{and} \quad e^{jx} = \cos x + j\sin x$$

we get

$$H(u,v) = \frac{1}{2}\left[\cos\frac{2\pi u}{N} + \cos\frac{2\pi v}{N}\right]$$

that is the filter transfer function in the frequency domain. ($H(u,v)$ is real because the filter is symmetric!)

b) $|H(0,0)| = 1$ and $|H|$ decreases as a function of distance from the origin. This is the characteristic of a lowpass filter. In fact this happens monotonically in the directions of u and v axis only; in the diagonal directions also the higher frequencies are emphasized:

$$\left| H(\pm\frac{N}{2}, \pm\frac{N}{2}) \right| = |-1| = 1$$

Look at the images!

c) The phase response can be calculated from $H(u,v)$. Because $\mathrm{Im}\{H(u,v)\} = 0$, then the phase response is 0 when $\mathrm{Re}\{H(u,v)\} > 0$, and π where $\mathrm{Re}\{H(u,v)\} < 0$:

$$\arg H(u,v) = \begin{cases} 0, & \text{if } (|u| + |v|)/N \leq 0.5 \\ \pi, & \text{if } (|u| + |v|)/N > 0.5 \end{cases}$$

Look at the images!

So H retains (but turns) "the chessboard" ($u = v = N/2$) but averages the horizontal and vertical lines in images.

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  To get started, type one of these commands: helpwin, helpdesk, or demo.
  For information on all of the MathWorks products, type tour.

>> u=-0.5:.05:0.5;
>> v=u;
>> [X,Y]=meshdom(u,v);
>> Z=0.5*(cos(2*pi.*X)+cos(2*pi.*Y));
>> subplot(3,2,1), mesh(u,v,Z);
>> subplot(3,2,2), mesh(u,v,abs(Z));
>> subplot(3,2,3), contour(u,v,Z);
>> subplot(3,2,4), contour(u,v,abs(Z));
>> subplot(3,2,5), mesh(u,v,atan2(imag(Z),real(Z)));
>> print -dps
```