T-61.5070 COMPUTER VISION, Exercise 9/08

1.

The Euler number (Sec. 6.3.1, p. 256), E, defines a topological property of an image: it is the difference between the number of connected components, C, and the number of holes, H.

$$E = C - H. \tag{1}$$

The Euler number is a rather general descriptor but as an additional feature it is often useful in characterizing regions in a scene. The Euler number is not affected by rotation, stretching, or scaling. This property is useful in character recognition, for instance. Fig. 1 shows four character images with different Euler numbers.

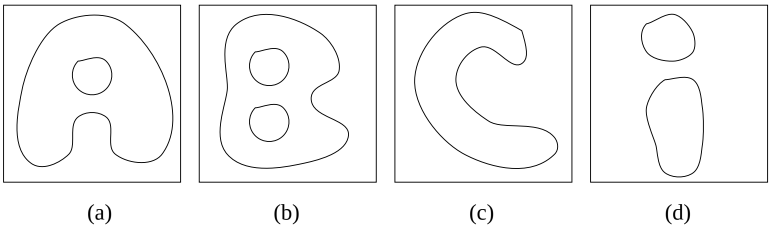


Figure 1: Topological description with the Euler number. (a) $E = 1-1 = 0$. (b) $E = 1-2 = -1$. (c) $E = 1 - 0 = 1$. (d) $E = 2 - 0 = 2$.

2.

Moments are defined by

$$m_{ij} = \sum_x \sum_y x^i y^j f(x, y). \tag{2}$$

A center of gravity is expressed by moments:

$$x_0 = \frac{\sum\sum x f(x,y)}{\sum\sum f(x,y)} = \frac{m_{10}}{m_{00}} \tag{3}$$

$$y_0 = \frac{\sum\sum y f(x,y)}{\sum\sum f(x,y)} = \frac{m_{01}}{m_{00}} \tag{4}$$

Central moments are defined by

$$\mu_{ij} = \sum_x \sum_y (x - x_0)^i (y - y_0)^j f(x, y). \tag{5}$$

They can be expressed as moments as shown in the following examples:

$$\begin{aligned}
&\mu_{00} = \sum\sum f(x,y) = m_{00} \\
&\mu_{01} = \sum\sum y f(x,y) - \sum\sum y_0 f(x,y) = m_{01} - (m_{01}/m_{00}) m_{00} = 0 \\
&\mu_{10} = \sum\sum x f(x,y) - \sum\sum x_0 f(x,y) = m_{10} - (m_{10}/m_{00}) m_{00} = 0 \\
&\mu_{11} = m_{11} - m_{01} m_{10}/m_{00} \\
&\mu_{20} = m_{20} - m_{10}^2/m_{00} \\
&\mu_{02} = m_{02} - m_{01}^2/m_{00}.
\end{aligned} \tag{6}$$

a)

Image 1

i	j	m_{ij}
0	0	$\sum\sum f(x,y)=14$
0	1	$\sum\sum yf(x,y)=38$
1	0	$\sum\sum xf(x,y)=32$
1	1	$\sum\sum xyf(x,y)=88$
2	0	$\sum\sum x^2f(x,y)=98$
0	2	$\sum\sum y^2f(x,y)=188$

Image 2

i	j	m_{ij}
0	0	$\sum\sum f(x,y)=14$
0	1	$\sum\sum yf(x,y)=43$
1	0	$\sum\sum xf(x,y)=31$
1	1	$\sum\sum xyf(x,y)=95$
2	0	$\sum\sum x^2f(x,y)=89$
0	2	$\sum\sum y^2f(x,y)=169$

b)

Image 1	Image 2
$x_0 = 2\frac{2}{7}$	$x_0 = 2\frac{3}{14}$
$y_0 = 2\frac{5}{7}$	$y_0 = 3\frac{1}{14}$

c)

Image 1

i	j	μ_{ij}
0	0	14
0	1	0
1	0	0
1	1	8/7
2	0	174/7
0	2	594/7

Image 2

i	j	μ_{ij}
0	0	14
0	1	0
1	0	0
1	1	-3/14
2	0	285/14
0	2	517/14

3.

PCA (principal component analysis) -transform is given by

$$\mathbf{y} = \mathbf{A}(\mathbf{x} - \mathbf{m_x}) \qquad \text{(the rows of } \mathbf{A} \text{ are the eigenvectors of } \mathbf{C_x}\text{)}$$

Mean:

$$\mathbf{m_x} = \frac{1}{M}\sum_{k=1}^{M}\mathbf{x}_k$$

Covariance matrix:

$$\mathbf{C_x} = \frac{1}{M}\sum_{k=1}^{M}\mathbf{x}_k\mathbf{x}_k^T - \mathbf{m_x}\mathbf{m_x}^T$$

The mean of the six sample points is now

$$\mathbf{m_x} = \frac{1}{6}\begin{pmatrix} -2-1+0+0+1+2 \\ 0+2+3+1+2+4 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

and the covariance matrix is

$$\mathbf{C_x} = \frac{1}{6}\begin{pmatrix} 4+1+0+0+1+4 & 0-2+0+0+2+8 \\ 0-2+0+0+2+8 & 0+4+9+1+4+16 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 5/3 & 4/3 \\ 4/3 & 5/3 \end{pmatrix}$$

Then we calculate the eigenvalues and eigenvectors of $\mathbf{C_x}$:

$$\mathbf{C_x}\mathbf{e}_i = \lambda_i\mathbf{e}_i \quad \Leftrightarrow \quad \begin{pmatrix} 5/3 & 4/3 \\ 4/3 & 5/3 \end{pmatrix}\begin{pmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \end{pmatrix} = \lambda_i\begin{pmatrix} \mathbf{e}_{i1} \\ \mathbf{e}_{i2} \end{pmatrix}$$

The solution is obtained when $|\mathbf{C_x} - \lambda\mathbf{I}| = 0$,

$$\begin{vmatrix} 5/3-\lambda & 4/3 \\ 4/3 & 5/3-\lambda \end{vmatrix} = \left(\frac{5}{3}\right)^2 - 2\cdot\frac{5}{3}\cdot\lambda + \lambda^2 - \left(\frac{4}{3}\right)^2 = \lambda^2 - \frac{10}{3}\lambda + 1 = 0 \Rightarrow \lambda_1 = 3 \text{ and } \lambda_2 = 1/3.$$

Next we calculate the eigenvectors

$$\Rightarrow \quad \mathbf{e}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

The transform matrix $\mathbf{A}$ is thus

$$\mathbf{A} = \begin{pmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

a) Transform 2-dim $\Rightarrow$ 1-dim: the transform matrix is the eigenvector that corresponds to the largest eigenvalue:

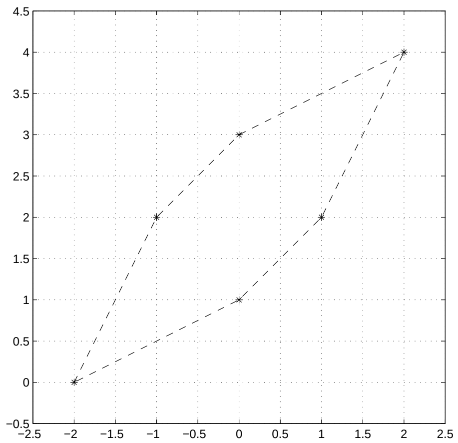
$$\mathbf{A} = \mathbf{e}_1^T = (1/\sqrt{2} \quad 1/\sqrt{2})$$

The average error is $R = \lambda_2 = 1/3$.
The transformed points are:

$$\mathbf{y}_1 = \mathbf{A}(\mathbf{x} - \mathbf{m_x}) = \mathbf{e}_1^T(\mathbf{x}_1 - \mathbf{m_x}) = (1/\sqrt{2} \quad 1/\sqrt{2}) \begin{pmatrix} -2-0 \\ 0-2 \end{pmatrix} = -2\sqrt{2}$$

$\mathbf{y}_2 = -\sqrt{2}/2$, $\mathbf{y}_3 = \sqrt{2}/2$, $\mathbf{y}_4 = -\sqrt{2}/2$, $\mathbf{y}_5 = \sqrt{2}/2$, $\mathbf{y}_6 = 2\sqrt{2}$.

b) The six sample points and the region spanned by them is shown below.



The elongatedness of the region can be obtained straight from the eigenvalues λ_1 and λ_2. Since they correspond to variances of the sample points along the principal components, the elongatedness is obtained from their ratio:

$$E = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_2}} = \sqrt{\frac{3}{1/3}} = 3.$$

So the elongatedness is now 3:1. (The square roots are needed to obtain the standard deviations from the variances.)