T-61.5070 COMPUTER VISION, Exercise 5/08

1.

We will need the following definitions of dilation and translation.

The dilation of the point set X by the structuring element B is defined as

$$X \oplus B = \{p \in \mathcal{E}^2 \mid p = x + b \text{ for some } x \in X \text{ and } b \in B\}.$$

The translation of the point set X by the vector h is defined by

$$X_h = \{p \in \mathcal{E}^2 \mid p = x + h \text{ for some } x \in X\}.$$

a) Prove that dilation is commutative, $X \oplus B = B \oplus X$.

Proof:

$$\begin{aligned} X \oplus B &= \{p \mid p = x + b \text{ for some } x \in X, b \in B\} \\ &= \{p \mid p = b + x \text{ for some } x \in X, b \in B\} = B \oplus X \end{aligned}$$

b) Prove that dilation is associative, $X \oplus (B \oplus D) = (X \oplus B) \oplus D$.

Proof:

$p \in X \oplus (B \oplus D)$ if and only if there exists $x \in X$, $b \in B$, and $d \in D$ such that $p = x + (b + d)$. $p \in (X \oplus B) \oplus D$ if and only if there exists $x \in X$, $b \in B$, and $d \in D$ such that $p = (x + b) + d$. But $x+(b+d) = (x+b)+d$ since addition is associative. Therefore, $X \oplus (B \oplus D) = (X \oplus B) \oplus D$.

c) Prove that dilation may also be expressed as a union of shifted point sets, $X \oplus B = \bigcup_{b \in B} X_b$.

Proof:

Suppose that $p \in X \oplus B$. Then for some $x \in X$ and $b \in B$, $p = x + b$. Hence, $p \in (X)_b$ and therefore $p \in \bigcup_{b \in B} X_b$.
Suppose $p \in \bigcup_{b \in B} X_b$. Then for some $b \in B$, $p \in (X)_b$. But $p \in (X)_b$ implies there exists an $x \in X$ such that $p = x + b$. Now by definition of dilation, $x \in X$ and $b \in B$, and $p = x + b$ imply $p \in X \oplus B$.

d) Prove that dilation is invariant to translation, $X_h \oplus B = (X \oplus B)_h$.

Proof:

$y \in X_h \oplus B$ if and only if for some $z \in X_h$ and $b \in B$, $y = z + b$. But $z \in X_h$ if and only if $z = x + h$ for some $x \in X$. Hence, $y = (x + h) + b = (x + b) + h$. Now by definition of dilation and translation $y \in (X \oplus B)_h$.

2.

Skeletonizing with thinning

The skeleton can be obtained with the thinning algorithm 6.8, p. 267.

The first image The set of region pixels R for the first image is

$$R = \begin{array}{|ccccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&1&1&1&1&1&1&0&0\\ 0&0&1&1&1&1&1&1&1&0&0\\ 0&0&1&1&1&1&1&1&1&0&0\\ 0&0&1&1&1&1&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

Define $H_i(R)$ as the inner boundary of R (border of image) and $H_o(R)$ as the outer boundary of R (border of background). Let $S(R)$ be a set of pixels from region R which have all their neighbors in 8-connectivity either from the inner boundary $H_i(R)$ or from the background. Assign $R_{old} = R$. Construct a region R_{new} which is the result of one-step thinning as follows

$$R_{new} = S(R_{old}) \cup [R_{old} - H_i(R_{old})] \cup [H_o(S(R_{old})) \cap R_{old}].$$

The sets $S(R_{old})$ and $H_o(S(R_{old})) \cap R_{old}$ are empty.

$$R_{new} = R_{old} - H_i(R_{old}) = \begin{array}{|ccccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

The new iteration gives the same image R_{new}. Therefore, R_{new} is a set of skeleton pixels of the region R.

The second image The set of region pixels R for the second image is

$$R = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&1&1&1&1&1&1&0&0\\ 0&1&1&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&1&1&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

Assign $R_{old} = R$. The sets $S(R_{old})$ and $H_o(S(R_{old})) \cap R_{old}$ are empty. Therefore, the region $R_{new} = R_{old} - H_i(R_{old})$.

$$R_{new} = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&1&1&1&1&1&1&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

Assign $R_{old} = R_{new}$ and continue.

$$S(R_{old}) = \begin{array}{|cccccccccc|}
\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\hline
\end{array}$$

$$R_{old} - H_i(R_{old}) = \begin{array}{|cccccccccc|}
\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\hline
\end{array}$$

$$H_o(S(R_{old})) = \begin{array}{|cccccccccc|}
\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&1&0&0\\
0&1&0&\mathbf{1}&0&0&\mathbf{1}&0&1&0\\
0&0&1&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\hline
\end{array}$$

The intersection of $H_o(S(R_{old}))$ and R_{old} is in bold. The union image is

$$R_{new} = \begin{array}{|cccccccccc|}
\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
\hline
\end{array}$$

The third iteration gives the same region. Therefore, R_{new} is a set of skeleton pixels of the region R.

Skeletonizing by maximal balls

The skeleton can be created using maximal balls as explained in the textbook section 13.5.2/11.5.2. For image X with structural elements B (usually B_4 or B_8 balls), the skeleton $S(X, B)$ is determined by the iterative set of operations

$$S(X, B) = \bigcup_{n=0}^{N} S_n(X, B),$$

where

$$\begin{aligned} S_n(X, B) &= (X \ominus nB) \setminus (X \ominus nB) \circ B \\ &= (X \ominus nB) \setminus ((X \ominus nB) \ominus B) \oplus B. \end{aligned}$$

The structural element B and nonnegative integer n define

$$nB = \underbrace{B \oplus B \oplus \cdots \oplus B}_{n \text{ times}}.$$

The value of N is the largest n with which $X \ominus nB$ does not produce an empty image. In this example we chose $B = B_4$, i.e.

$$B : \begin{array}{|ccc|}\hline 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ \hline\end{array} \; 2B : \begin{array}{|ccccc|}\hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline\end{array} \; 3B : \begin{array}{|ccccccc|}\hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline\end{array}$$

The first image Start iterating the first image:

$n = 0 :\ S_0(X, B) = X \setminus (X \ominus B) \oplus B$

$$X \ominus B = \begin{array}{|ccccccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array}$$

$$(X \ominus B) \oplus B = \begin{array}{|ccccccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array}$$

$$S_0(X, B) = \begin{array}{|ccccccccccc|}\hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline\end{array}$$

$n = 1: \ S_1(X, B) = X \ominus B \setminus ((X \ominus B) \ominus B) \oplus B$

$$X \ominus B = \begin{array}{|ccccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$$((X \ominus B) \ominus B) \oplus B = \begin{array}{|ccccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$$S_1(X, B) = \begin{array}{|ccccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$n = 2:$ The set $X \ominus 2B$ is empty. Therefore, $S_2(X, B)$ is also empty and $N = 1$.

The skeleton of the first image is

$$S(X, B) = S_0(X, B) \cup S_1(X, B) =$$

$$\begin{array}{|ccccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&0&1&1&1&1&1&0&0&0\\ 0&0&1&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

The second image

$n = 0: \ S_0(X, B) = X \setminus (X \ominus B) \oplus B$

The opening of X using B produces X. Therefore, $S_0(X, B)$ is an empty set.

$n = 1: \ S_1(X, B) = X \ominus B \setminus ((X \ominus B) \ominus B) \oplus B$

$$X \ominus B = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&1&1&1&1&1&1&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$$((X \ominus B) \ominus B) \oplus B = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&1&1&1&1&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$$S_1(X, B) = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$n = 2:\ S_2(X, B) = X \ominus 2B \setminus ((X \ominus 2B) \ominus B) \oplus B$

$$X \ominus 2B = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$$((X \ominus 2B) \ominus B) \oplus B = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$$S_2(X, B) = \begin{array}{|cccccccccc|}\hline 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \hline\end{array}$$

$n = 3:$ The set $X \ominus 3B$ is empty. Therefore, $S_3(X, B)$ is empty and $N = 2$.

The skeleton of the second image is

$$S(X,B) = S_0(X,B) \cup S_1(X,B) \cup S_2(X,B) =$$

$$\begin{array}{|cccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}$$

Final skeletons produced

Original image

$$\begin{array}{|ccccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&1&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}
\qquad
\begin{array}{|cccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&1&0&0&0&0\\
0&0&1&1&1&1&1&1&0&0\\
0&1&1&1&1&1&1&1&1&0\\
0&0&1&1&1&1&1&1&0&0\\
0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}$$

Thinning algorithm

$$\begin{array}{|ccccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&0&0&0\\
0&0&0&1&1&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}
\qquad
\begin{array}{|cccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}$$

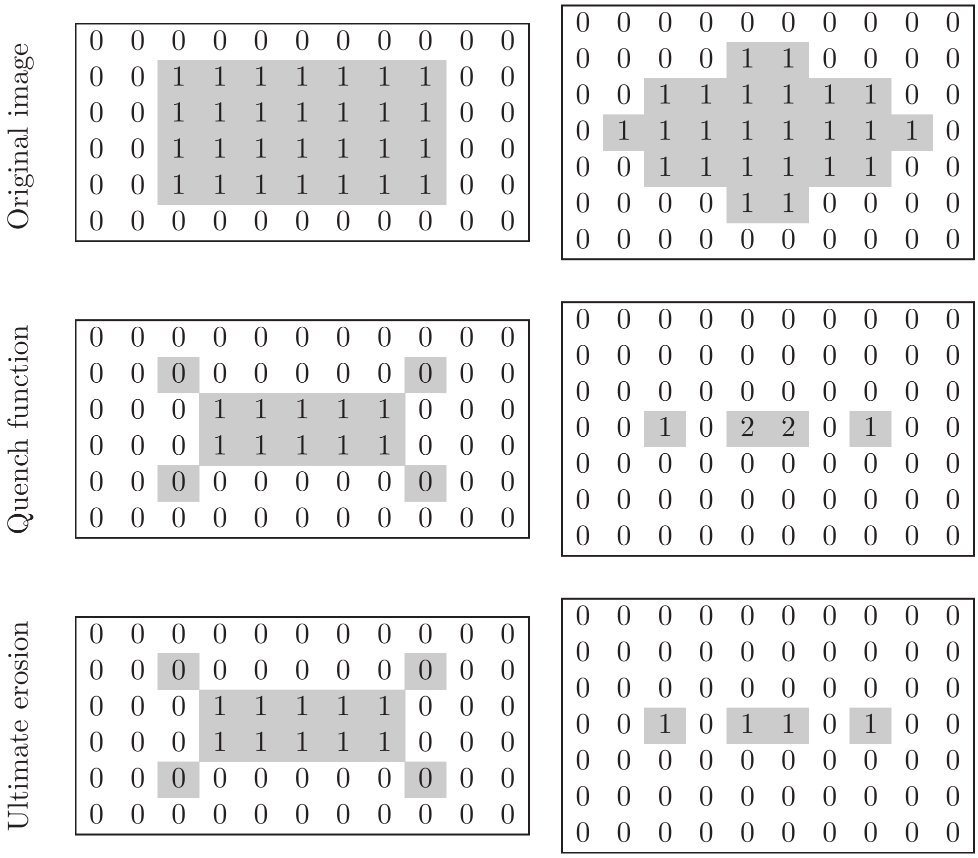
Maximal balls

$$\begin{array}{|ccccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&1&1&1&1&1&0&0&0\\
0&0&0&1&1&1&1&1&0&0&0\\
0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}
\qquad
\begin{array}{|cccccccccc|}\hline
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\ \hline
\end{array}$$

In comparison, the maximal balls algorithm produced a better skeleton for the first image, because it also included the “lines” for the corners. For the second image, the thinning algorithm was better, because the result from maximal balls was not contiguous, i.e. the transformation was not homotopic.

3.

The quench function and ultimate erosions are described in the book (section 11.5.4/13.5.4). Every point p of the skeleton by maximal balls has an associated ball which was a maximum ball at that point when performing the algorithm. The quench function $q_X(p)$ is simply the radius of that ball at each point. We can define it to be zero outside of the skeleton. Using this function we can recreate the original object by placing a ball of the radius given by $q_X(p)$ at each point. For the skeletons created by thinning there is no direct connection to the quench function because there are no maximal balls. The appropriate type of ball in this case is B_4 since the skeletons were created with that.



Original image

0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	1	1	1	1	0	0
0	0	1	1	1	1	1	1	1	0	0
0	0	1	1	1	1	1	1	1	0	0
0	0	1	1	1	1	1	1	1	0	0
0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	0	0	0	0
0	0	1	1	1	1	1	1	0	0
0	1	1	1	1	1	1	1	1	0
0	0	1	1	1	1	1	1	0	0
0	0	0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Quench function

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	0	2	2	0	1	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

Ultimate erosion

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	0	0	0
0	0	0	1	1	1	1	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	1	0	1	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

The ultimate erosion of an image is the set of regional maxima of the quench function. The distinction between regional and local maxima are explained in the textbook (section 11.5.4/13.5.4). The regional maximum is a connected set of pixels M with the same value (e.g. grayscale value) h, such that every neighboring pixel of M has strictly lower value than h. Here we have used 4-neighborhoods.