Covering problems

10. Covering problems

- Given a set of concepts (rules etc.) that apply to examples (rows of data etc.)
- The concept *covers* the examples
- How to find good small collections of concepts?
- Not all concepts satisfying certain conditions

Example: rows and attributes

- Given a 0-1 matrix
- Set cover: Find a collection Z of variables such that for every row t there is at least one variable $A \in Z$ such that $t(A) = 1$
- Best collection: Find a set Z of variables with $|Z| = k$ such that there are as many rows t as possible such that $t(A) = 1$ for some $A \in Z$.

Prototype problems

- **Set cover problem**: find a small set of concepts such that all examples are covered by some concept in the set
- **Best collection problem**: find a set of size k of concepts that covers as many examples as possible
- Both problems are NP-complete
- Simple approximation algorithms with provable properties

Set cover problem

- Given a universe $X = p_1, \ldots, p_n$
- Sets $S_1, S_2, \ldots, S_m \subseteq X$, $\cup S_i = X$
- $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$.
- Question: find the smallest number of set from $\mathcal{F}$ whose union is X
- I.e., find a smallest subcollection $\mathcal{C} \subseteq \mathcal{F}$ such that

$$\cup_{S \in \mathcal{C}} S = X.$$

- NP-complete (what does it mean?)

Trivial algorithm

- Try all subcollections of $\mathcal{F}$
- Select the smalles one that covers X
- Running time $\mathcal{O}(2^{|\mathcal{F}|}|X|)$
- Too slow

Greedy algorithm for set cover

- Select first the largest set S
- Remove the elements of S from X
- Recompute the sizes of the sets
- Go back to the first step

As an algorithm

1. $U = X$;
2. $\mathcal{C} = \emptyset$;
3. While U is not empty do
   - For all $S \in \mathcal{F}$ let $a_S = |Y_i \cap U|$
   - Let S be such that a_S is maximal;
   - $\mathcal{C} = \mathcal{C} \cup \{S\}$
   - $U = U \setminus S$

How can this go wrong?

- No global consideration of how good or bad the set will be

Weighted version

- Each set $S \in \mathcal{F}$ has a cost $c(S)$
- Compute the number of elements per unit cost: $(S \cap U)/c(S)$
- At each step, select the S for which this is maximal

Running time of the algorithm

- Polynomial in $|X|$ and $|\mathcal{F}|$
- At most $min(|X|, |\mathcal{F}|)$ iterations of the loop
- Loop body takes time $\mathcal{O}(|X||\mathcal{F}|)$
- Running time $\mathcal{O}(|X||\mathcal{F}|min(|X|, |\mathcal{F}|))$
- Can be implemented in linear time $\mathcal{O}(\sum_{S \in \mathcal{F}} |S|)$

Related problems

- Given a graph $G = (V, E)$
- Independent set: find the largest set V' of vertices such that there is no edge between any two vertices in V'
- Glique: find the largest set of vertices V' such that all pairs of vertices of V' are connected by an edge
- Clique in G is an independent set in $\overline{G} = (V, V \times V \setminus E)$

Related problems

- Given a graph $G = (V, E)$
- **Vertex cover**: find the smallest subset $V' \subseteq V$ such that for each edge $(u, v) \in E$ we have $u \in V'$ or $v \in V'$

Approximation algorithms

- Consider a minimization problem
- Instance I, cost of optimal solution $\alpha^*(I)$, approximate solution with cost $\alpha(I)$
- The algorithm has approximation ratio $c(n)$, if for all instances of size at most n we have $\alpha(I) \leq c(n)\alpha^*(I)$
- The greedy algorithm has approximation ratio $\mathcal{O}(log\ n)$ (i.e., d log n for some d).

Approximation algorithm for vertex cover

- $C = \emptyset$;
- Select a random edge (u, v)
- $C = C \cup \{u, v\}$;
- Remove all edges that are incident either with u or with v
- Repeat until no edges remain

Approximation guarantee

- The result is a vertex cover (why?)
- No two selected edges share an endpoint
- For any edge (u, v) at least one of u and v has to belong to any vertex cover
- For any edge (u, v) at least one of u and v has to belong to the optimal vertex cover
- Thus $\alpha(G) \leq 2\alpha^*(G)$ for all G

Analysis of the greedy approximation algorithm for set cover

- $H(d) = \sum_{i=1}^{d} 1/i$: the ith harmonic number
- Greedy approximation algorithm has approximation ratio $H(s)$, where s is the size of the largest set in $\mathcal{F}$
- (Trivial bound; s)
- $H(s) \approx ln\ s$, i.e., the bound is quite good $(s \leq |X|)$

Proof.

- Source: Cormen et al., Introduction to Algorithms
- Optimal set cover $\mathcal{C}^*$ and the cover $\mathcal{C}$ produced by the greedy algorithm
- S_i: the ith set selected by an algorithm
- Cost of 1 (counting the number of sets)
- Spread this among the elements covered for the first time by S_i
- Cost to item x is $c_x = (|S_i \setminus \ (S_1 \cup \ldots S_{i-1})|)^{-1}$
- Costs by set S_i sum up to one
- $|\mathcal{C}| = \sum_{x \in X} c_x$

$\mathcal{C}^*$ covers X

$$|\mathcal{C}| = \sum_{x \in X} c_x \leq \sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x$$

For any set $S \in \mathcal{F}$ we have (proof separate)

$$\sum_{x \in S} c_x \leq H(|S|)$$

Thus

$$|\mathcal{C}| \leq \sum_{S \in \mathcal{C}^*} H(|S|)$$

and hence i

$$|\mathcal{C}| \leq \mathcal{C}^* H(s)$$

where $s = max\{|S| : S \in \mathcal{F}\}$

Best collection problem

- Given some concepts (rules etc.)
- Find the best collection of k rules
- For example, find the k sets whose union has maximum size
- Maximization problem, quality $f(\mathcal{C})$ of the result $=$ number of elements in $\cup_{S \in \mathcal{C}} S$
- Simple approximation algorithm has bound

$$\alpha \geq \frac{e-1}{e} \alpha^*$$

Submodular functions

- The result is very general
- Concepts $\mathcal{F}$, task fo find the best subcollection $\mathcal{C}^* \subseteq \mathcal{F}$ of size k
- Solution function f should satisfy for all $\mathcal{C}$

$$f(\mathcal{C}) \geq 0$$

and for all $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{F}$ and $S \in \mathcal{F}$

$$f(\mathcal{C} \cup \{S\}) - f(\mathcal{C}) \geq f(\mathcal{D} \cup \{S\}) - f(\mathcal{D})$$

i.e., the improvement obtained by adding S may not increase when moving to a larger solution

- f is submodular; the result holds for all such functions

Greedy approximation algorithm

- $\mathcal{C} = \emptyset$
- Gain of S in the context of $\mathcal{C}$ is $f(\mathcal{C} \cup \{S\}) - f(\mathcal{C})$
- Select the concept S that has the highest gain
- $\mathcal{C} := \mathcal{C} \cup \{S\}$
- Repeat until $\mathcal{C}$ has k elements

Basic theorem

Let $\mathcal{C}_k^*$ be the optimal set of k concepts

Let $\mathcal{C}_i$ be the ith set formed by the greedy algorithm.

Assume

$$f(\mathcal{C}_i) - f(\mathcal{C}_{i-1}) \geq \frac{1}{k}(f(\mathcal{C}_k^*) - f(\mathcal{C}_{i-1}))$$

Then

$$f(\mathcal{C}_k) \geq \frac{e-1}{e} f(\mathcal{C}_k^*)$$

Proof. Separate.

Why does the assumption hold?

$$f(\mathcal{C}_i) - f(\mathcal{C}_{i-1}) \geq \frac{1}{k}(f(\mathcal{C}_k^*) - f(\mathcal{C}_{i-1}))$$

f is submodular

the greedy approximation algorithm

The concept $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ is the one that maximizes the gain.

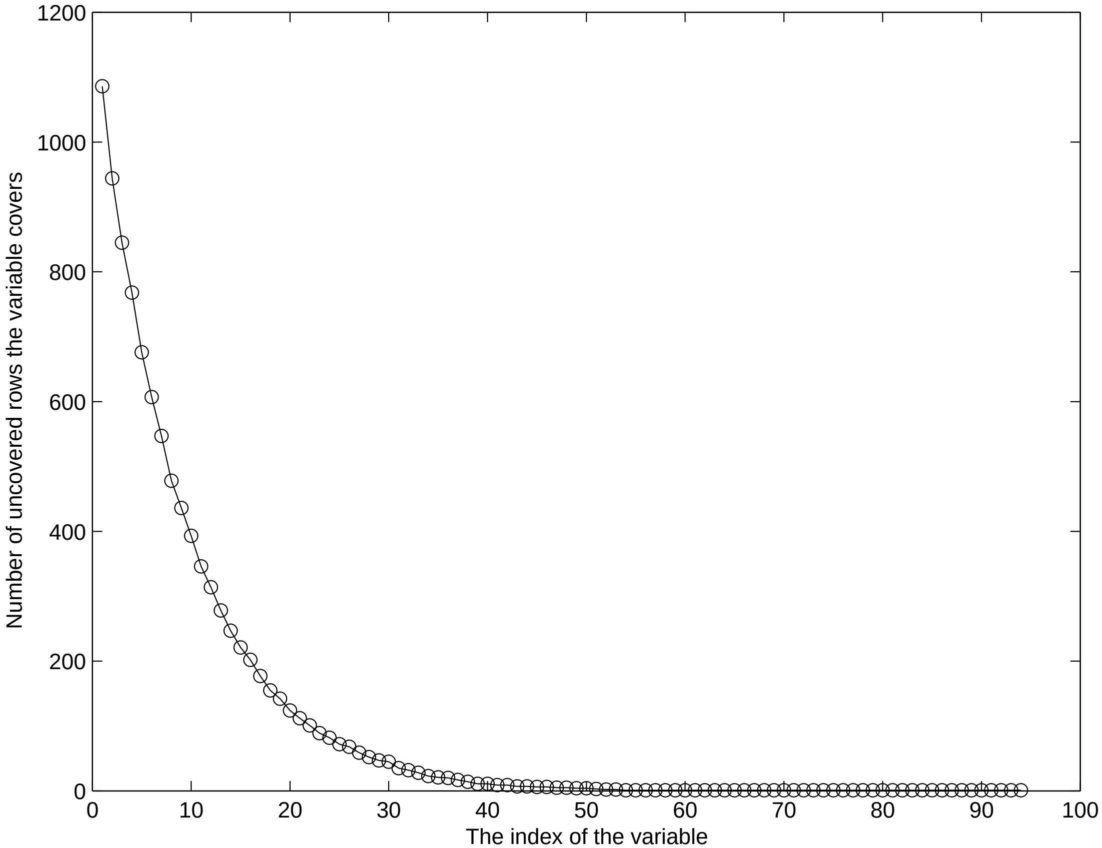
(Something open here.)

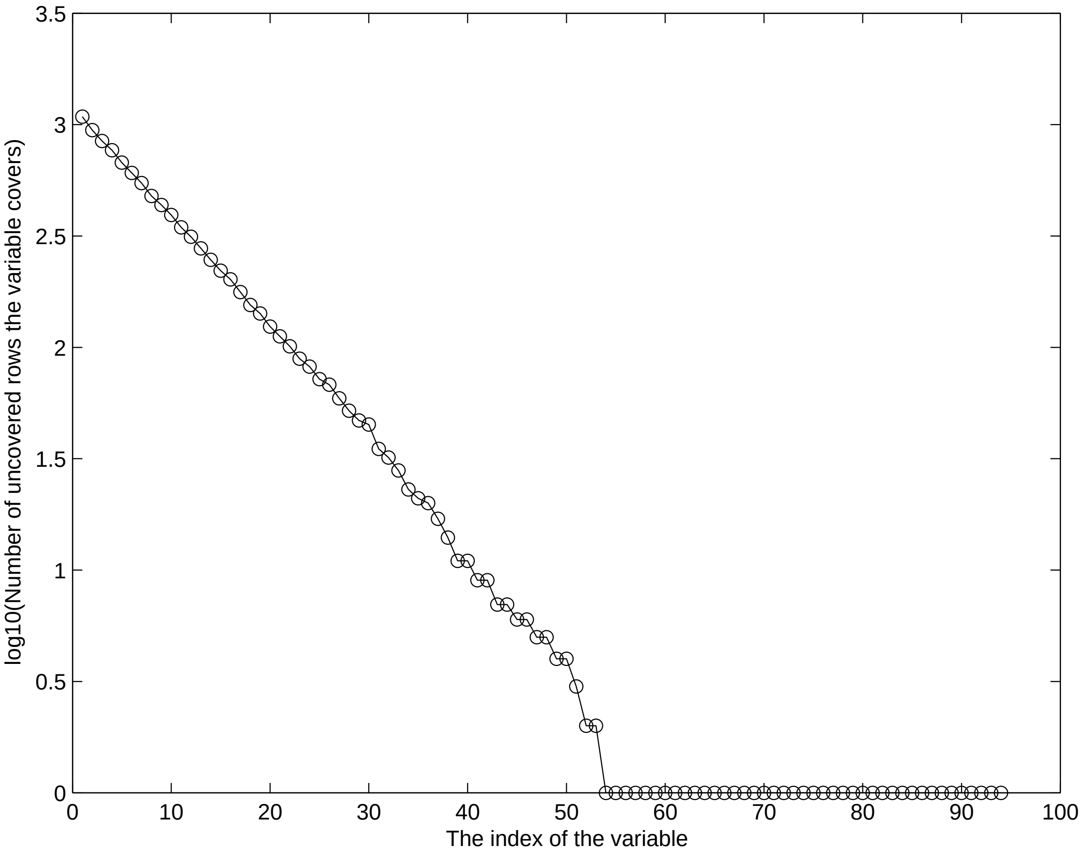
Applications

- Which functions are submodular?
- The concepts should not have interaction
- Variable selection?

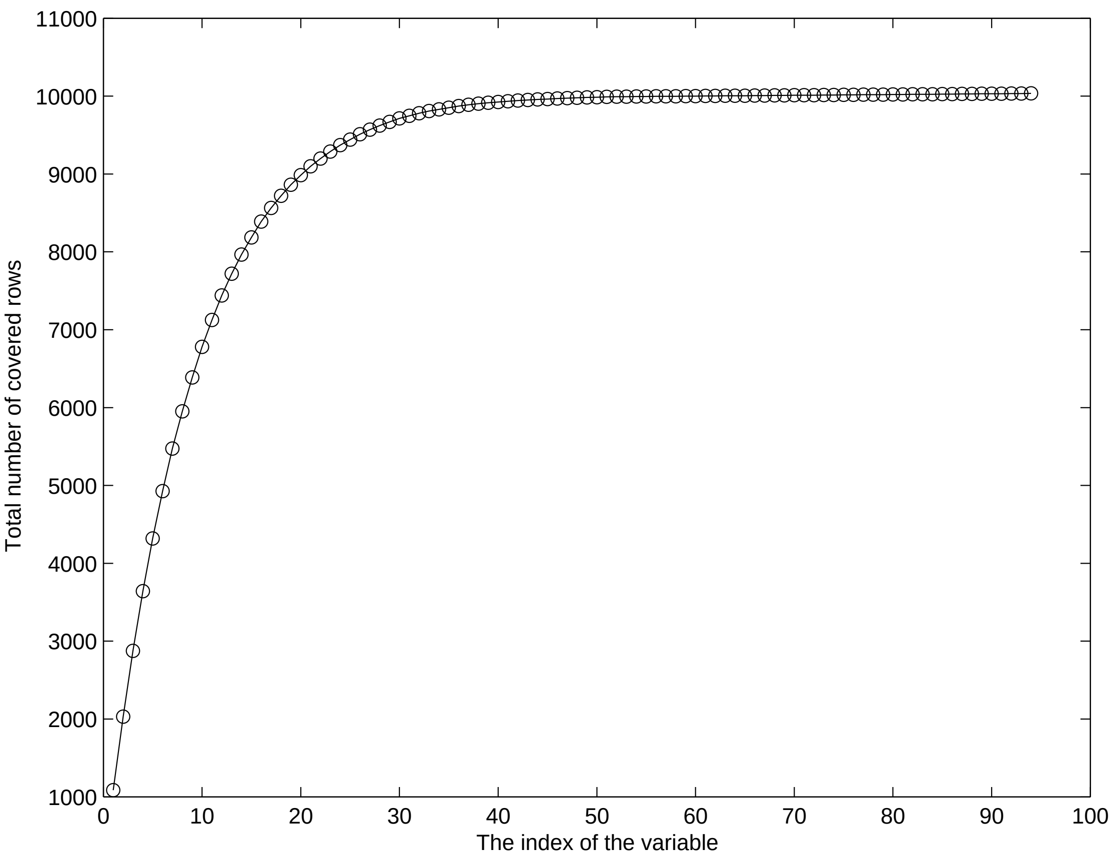
Some experiments

10000 rows, 100 variables, each true with probability 0.1





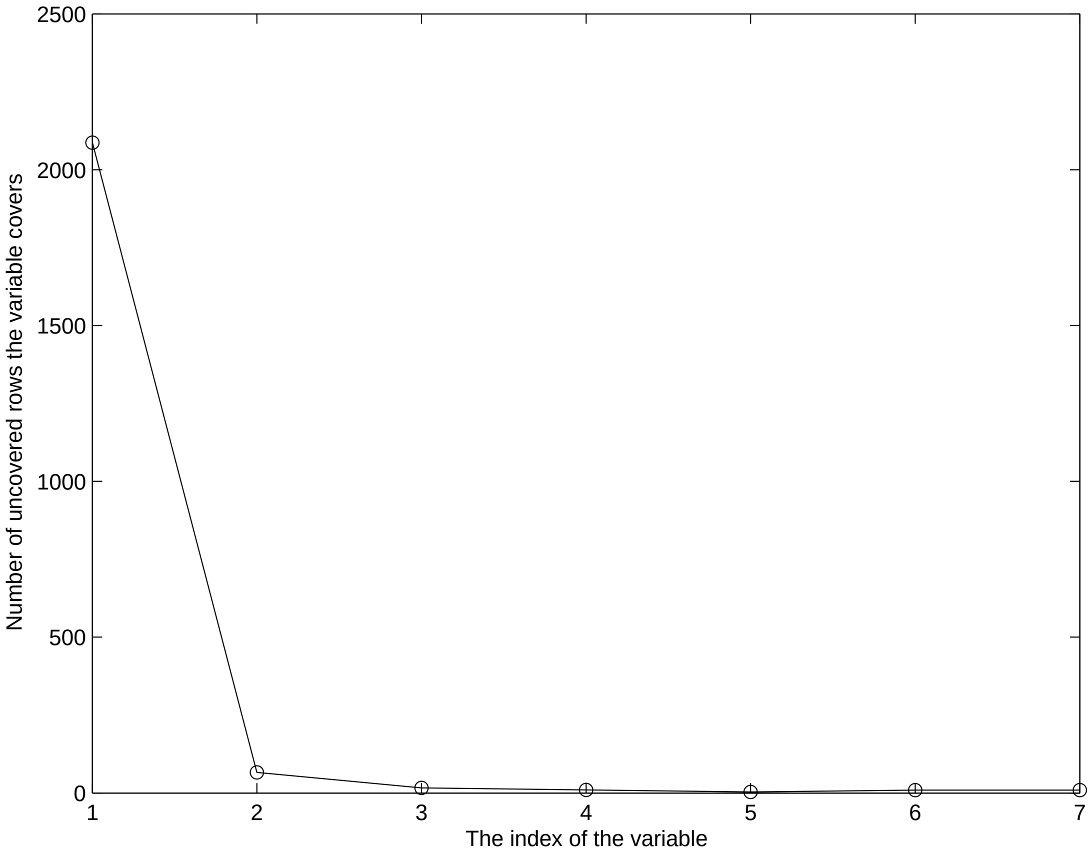
log10(Number of uncovered rows the variable covers)
The index of the variable
0
0.5
1
1.5
2
2.5
3
3.5
0
10
20
30
40
50
60
70
80
90
100

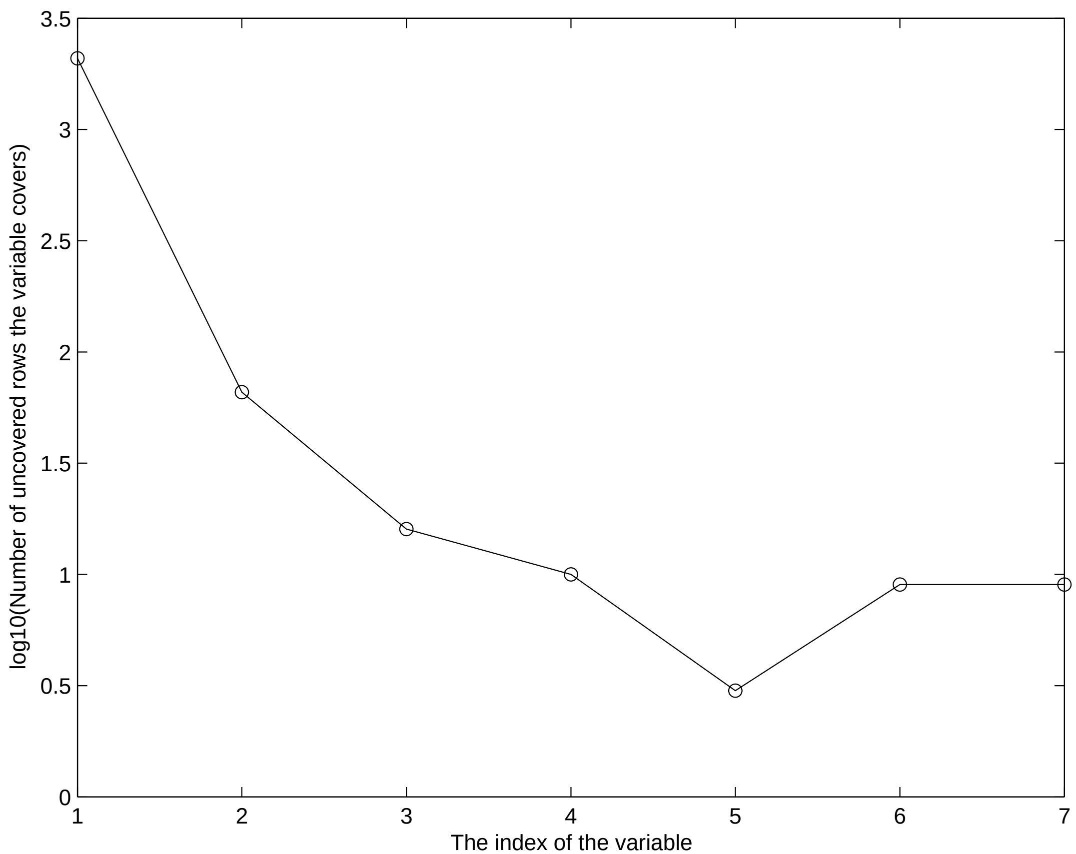


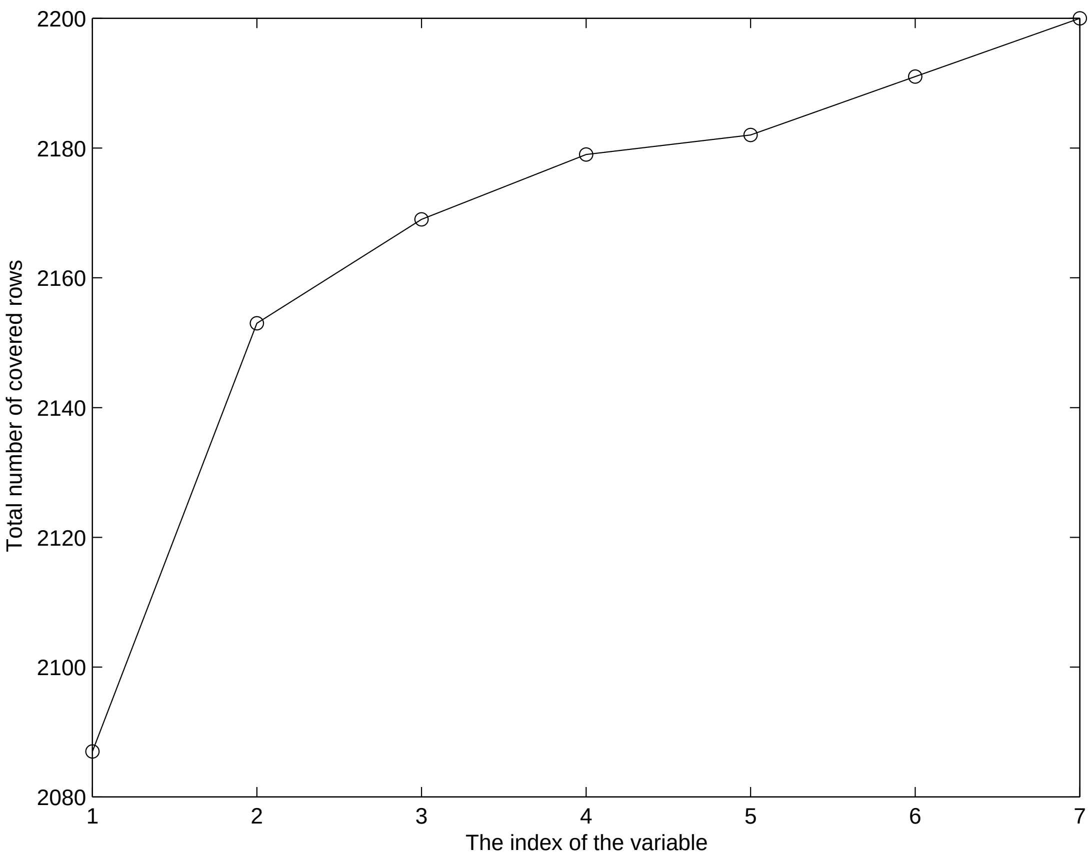
Total number of covered rows
The index of the variable
11000
10000
9000
8000
7000
6000
5000
4000
3000
2000
1000
0
10
20
30
40
50
60
70
80
90
100

Other dataset

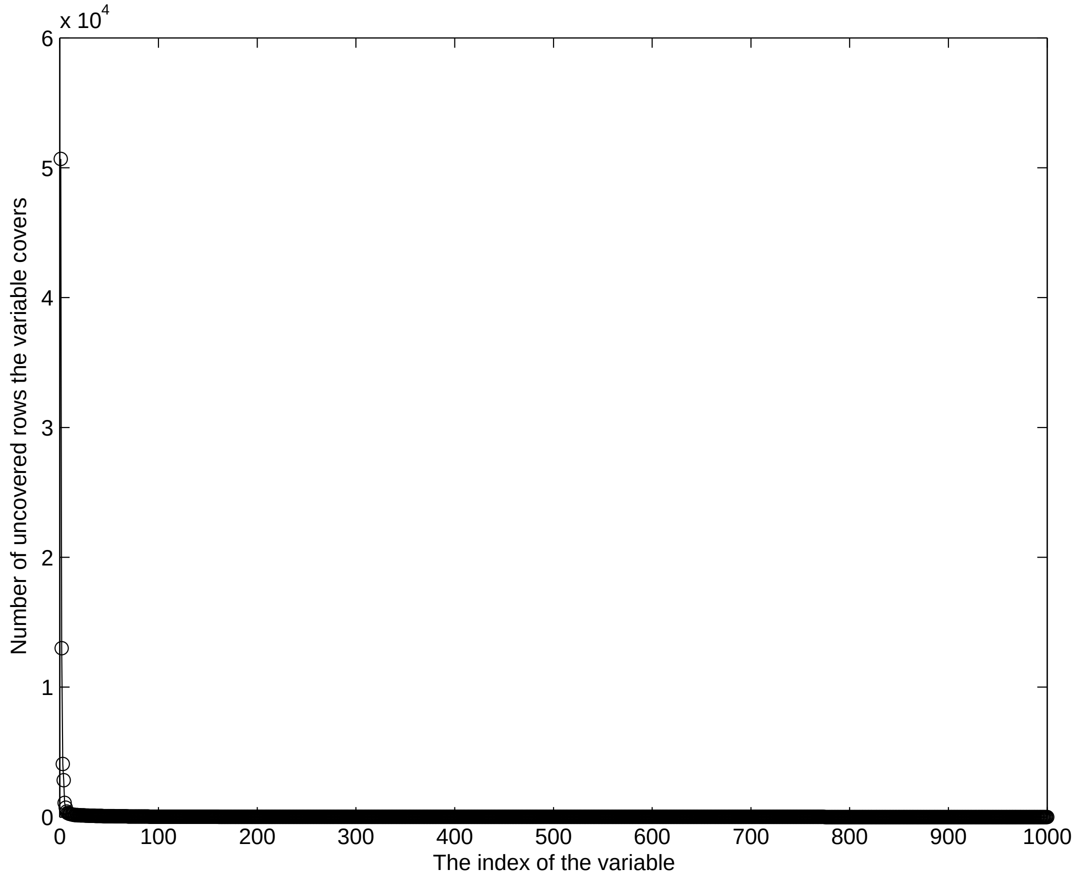
European mammals: 2183 rows, 124 variables (presence/absence in a grid)

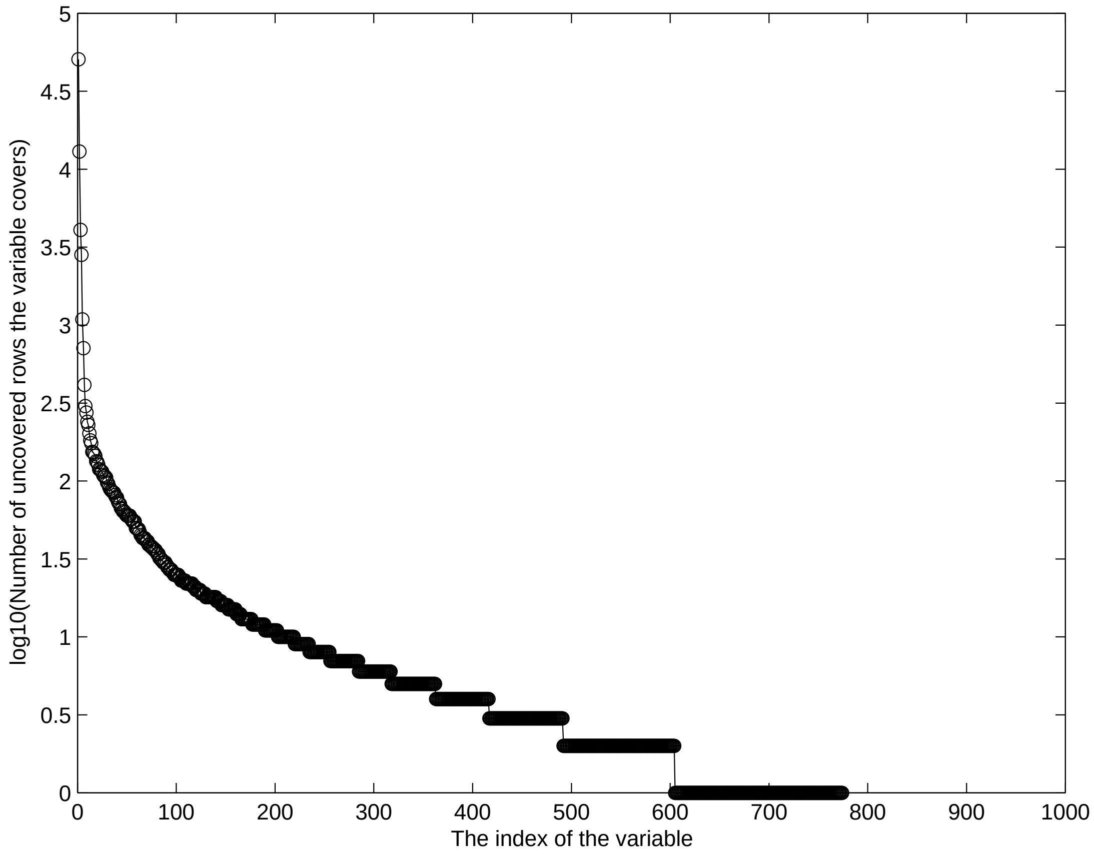



3.5
3
2.5
2
1.5
1
0.5
0
1
2
3
4
5
6
7
The index of the variable
log10(Number of uncovered rows the variable covers)

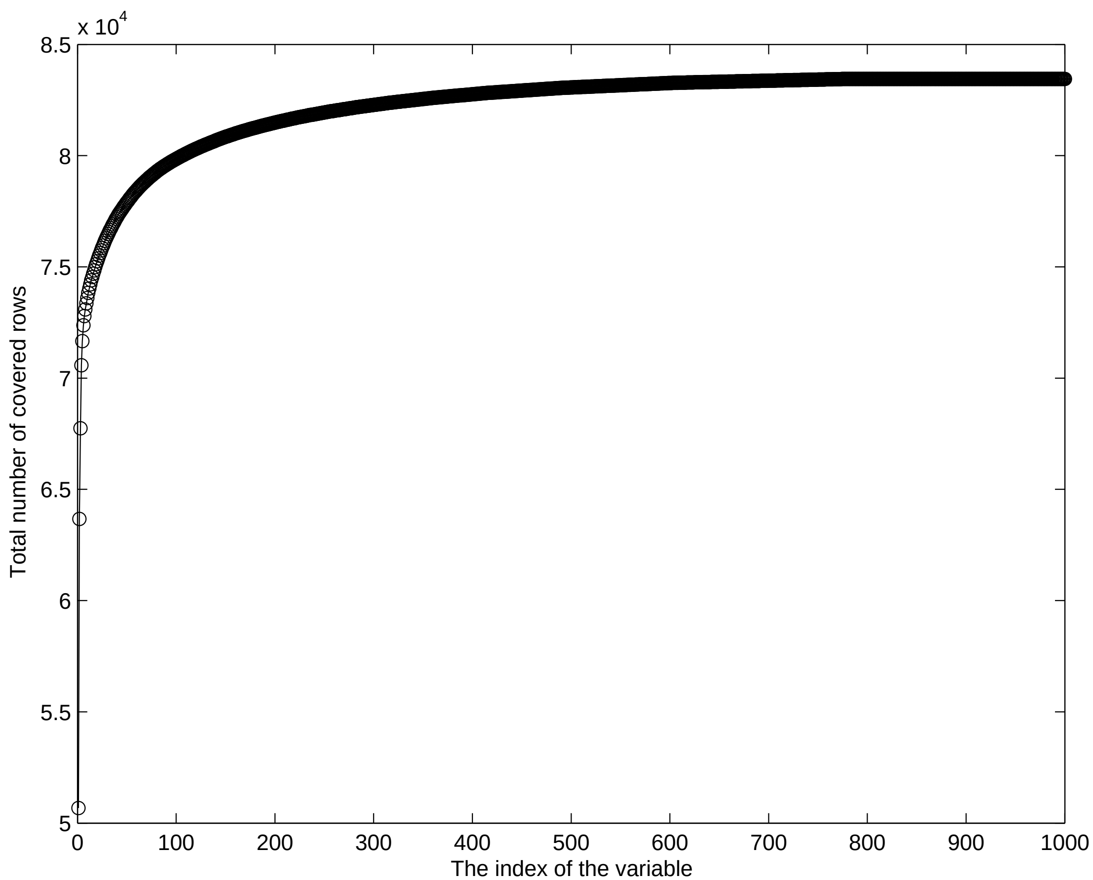

2200
2180
2160
2140
2120
2100
2080
Total number of covered rows
1
2
3
4
5
6
7
The index of the variable

Retail data, 88162 rows, 1000 columns





log10(Number of uncovered rows the variable covers)
The index of the variable
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
100
200
300
400
500
600
700
800
900
1000


x 10^4
8.5
8
7.5
7
6.5
6
5.5
5
0
100
200
300
400
500
600
700
800
900
1000
The index of the variable
Total number of covered rows