Covering problems

10. Covering problems

- Given a set of concepts (rules etc.) that apply to examples (rows of data etc.)
- The concept *covers* the examples
- How to find good small collections of concepts?
- Not all concepts satisfying certain conditions

Example: rows and attributes

- Given a 0-1 matrix
- Set cover: Find a collection Z of variables such that for every row t there is at least one variable $A \in Z$ such that $t(A) = 1$
- Best collection: Find a set Z of variables with $|Z| = k$ such that there are as many rows t as possible such that $t(A) = 1$ for some $A \in Z$.

Prototype problems

- **Set cover problem**: find a small set of concepts such that all examples are covered by some concept in the set
- **Best collection problem**: find a set of size k of concepts that covers as many examples as possible
- Both problems are NP-complete
- Simple approximation algorithms with provable properties

Set cover problem

- Given a universe $X = p_1, \ldots, p_n$
- Sets $S_1, S_2, \ldots, S_m \subseteq X$, $\cup S_i = X$
- $\mathcal{F} = \{S_1, S_2, \ldots, S_m\}$.
- Question: find the smallest number of set from $\mathcal{F}$ whose union is X
- I.e., find a smallest subcollection $\mathcal{C} \subseteq \mathcal{F}$ such that

$$\cup_{S \in \mathcal{C}} S = X.$$

- NP-complete (what does it mean?)

Trivial algorithm

- Try all subcollections of $\mathcal{F}$
- Select the smalles one that covers X
- Running time $\mathcal{O}(2^{|\mathcal{F}|}|X|)$
- Too slow

Greedy algorithm for set cover

- Select first the largest set S
- Remove the elements of S from X
- Recompute the sizes of the sets
- Go back to the first step

As an algorithm

1. $U = X$;
2. $\mathcal{C} = \emptyset$;
3. While U is not empty do
   - For all $S \in \mathcal{F}$ let $a_S = |Y_i \cap U|$
   - Let S be such that a_S is maximal;
   - $\mathcal{C} = \mathcal{C} \cup \{S\}$
   - $U = U \backslash S$

How can this go wrong?

- No global consideration of how good or bad the set will be

Weighted version

- Each set $S \in \mathcal{F}$ has a cost $c(S)$
- Compute the number of elements per unit cost: $(S \cap U)/c(S)$
- At each step, select the S for which this is maximal

Running time of the algorithm

- Polynomial in $|X|$ and $|\mathcal{F}|$
- At most $min(|X|, |\mathcal{F}|)$ iterations of the loop
- Loop body takes time $\mathcal{O}(|X||\mathcal{F}|)$
- Running time $\mathcal{O}(|X||\mathcal{F}|min(|X|, |\mathcal{F}|))$
- Can be implemented in linear time $\mathcal{O}(\sum_{S\in\mathcal{F}} |S|)$

Related problems

- Given a graph $G = (V, E)$
- Independent set: find the largest set V' of vertices such that there is no edge between any two vertices in V'
- Glique: find the largest set of vertices V' such that all pairs of vertices of V' are connected by an edge
- Clique in G is an independent set in $\overline{G} = (V, V \times V \backslash E)$

Related problems

- Given a graph $G = (V, E)$
- **Vertex cover**: find the smallest subset $V' \subseteq V$ such that for each edge $(u, v) \in E$ we have $u \in V'$ or $v \in V'$

Approximation algorithms

- Consider a minimization problem
- Instance I, cost of optimal solution $\alpha^*(I)$, approximate solution with cost $\alpha(I)$
- The algorithm has approximation ratio $c(n)$, if for all instances of size at most n we have $\alpha(I) \leq c(n)\alpha^*(I)$
- The greedy algorithm has approximation ratio $\mathcal{O}(log\ n)$ (i.e., d log n for some d).

Approximation algorithm for vertex cover

- $C = \emptyset$;
- Select a random edge (u, v)
- $C = C \cup \{u, v\}$;
- Remove all edges that are incident either with u or with v
- Repeat until no edges remain

Approximation guarantee

- The result is a vertex cover (why?)
- No two selected edges share an endpoint
- For any edge (u, v) at least one of u and v has to belong to any vertex cover
- For any edge (u, v) at least one of u and v has to belong to the optimal vertex cover
- Thus $\alpha(G) \leq 2\alpha^*(G)$ for all G

Analysis of the greedy approximation algorithm for set cover

- $H(d) = \sum_{i=1}^{d} 1/i$: the ith harmonic number
- Greedy approximation algorithm has approximation ratio $H(s)$, where s is the size of the largest set in $\mathcal{F}$
- (Trivial bound; s)
- $H(s) \approx ln\ s$, i.e., the bound is quite good $(s \leq |X|)$

Proof.

- Source: Cormen et al., Introduction to Algorithms
- Optimal set cover $\mathcal{C}^*$ and the cover $\mathcal{C}$ produced by the greedy algorithm
- S_i: the ith set selected by an algorithm
- Cost of 1 (counting the number of sets)
- Spread this among the elements covered for the first time by S_i
- Cost to item x is $c_x = (|S_i \setminus\ (S_1 \cup \ldots S_{i-1})|)^{-1}$
- Costs by set S_i sum up to one
- $|\mathcal{C}| = \sum_{x \in X} c_x$

$\mathcal{C}^*$ covers X

$$|\mathcal{C}| = \sum_{x \in X} c_x \leq \sum_{S \in \mathcal{C}^*} \sum_{x \in S} c_x$$

For any set $S \in \mathcal{F}$ we have (proof separate)

$$\sum_{x \in S} c_x \leq H(|S|)$$

Thus

$$|\mathcal{C}| \leq \sum_{S \in \mathcal{C}^*} H(|S|)$$

and hence i

$$|\mathcal{C}| \leq \mathcal{C}^* H(s)$$

where $s = max\{|S| : S \in \mathcal{F}\}$

Best collection problem

- Given some concepts (rules etc.)
- Find the best collection of k rules
- For example, find the k sets whose union has maximum size
- Maximization problem, quality $f(\mathcal{C})$ of the result = number of elements in $\cup_{S \in \mathcal{C}} S$
- Simple approximation algorithm has bound

$$\alpha \geq \frac{e-1}{e} \alpha^*$$

Submodular functions

- The result is very general
- Concepts $\mathcal{F}$, task fo find the best subcollection $\mathcal{C}^* \subseteq \mathcal{F}$ of size k
- Solution function f should satisfy for all $\mathcal{C}$

$$f(\mathcal{C}) \geq 0$$

  and for all $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{F}$ and $S \in \mathcal{F}$

$$f(\mathcal{C} \cup \{S\}) - f(\mathcal{C}) \geq f(\mathcal{D} \cup \{S\}) - f(\mathcal{D})$$

  i.e., the improvement obtained by adding S may not increase when moving to a larger solution
- f is submodular; the result holds for all such functions

Greedy approximation algorithm

- $\mathcal{C} = \emptyset$
- Gain of S in the context of $\mathcal{C}$ is $f(\mathcal{C} \cup \{S\}) - f(\mathcal{C})$
- Select the concept S that has the highest gain
- $\mathcal{C} := \mathcal{C} \cup \{S\}$
- Repeat until $\mathcal{C}$ has k elements

Basic theorem

Let $\mathcal{C}_k^*$ be the optimal set of k concepts

Let $\mathcal{C}_i$ be the ith set formed by the greedy algorithm.

Assume

$$f(\mathcal{C}_i) - f(\mathcal{C}_{i-1}) \geq \frac{1}{k}(f(\mathcal{C}_k^*) - f(\mathcal{C}_{i-1}))$$

Then

$$f(\mathcal{C}_k) \geq \frac{e-1}{e} f(\mathcal{C}_k^*)$$

Proof. Separate.

Why does the assumption hold?

$$f(\mathcal{C}_i) - f(\mathcal{C}_{i-1}) \geq \frac{1}{k}(f(\mathcal{C}_k^*) - f(\mathcal{C}_{i-1}))$$

f is submodular

the greedy approximation algorithm

The concept $\mathcal{C}_i \setminus \mathcal{C}_{i-1}$ is the one that maximizes the gain.

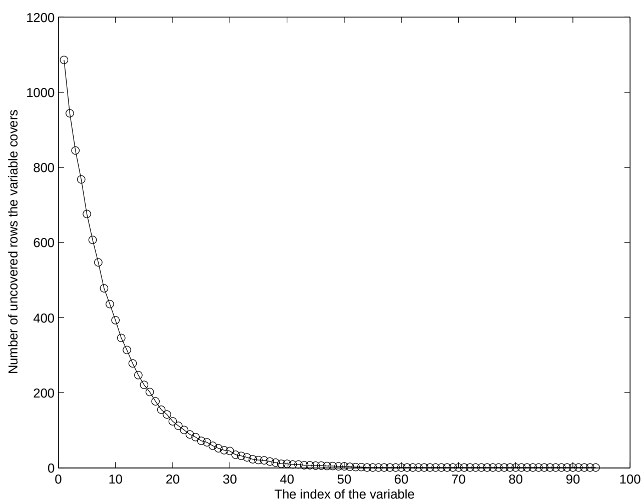
(Something open here.)

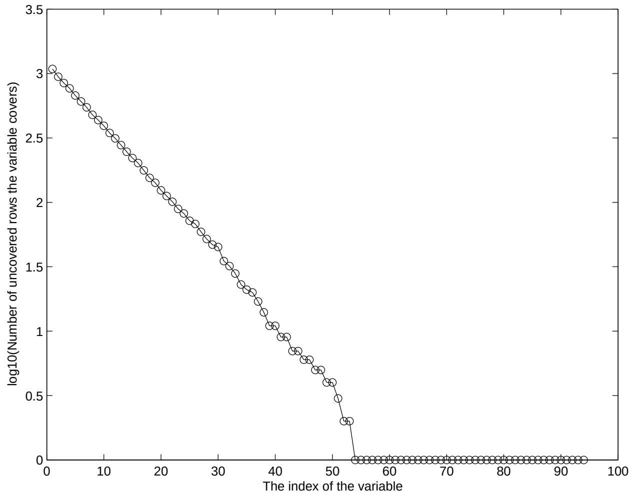
Applications

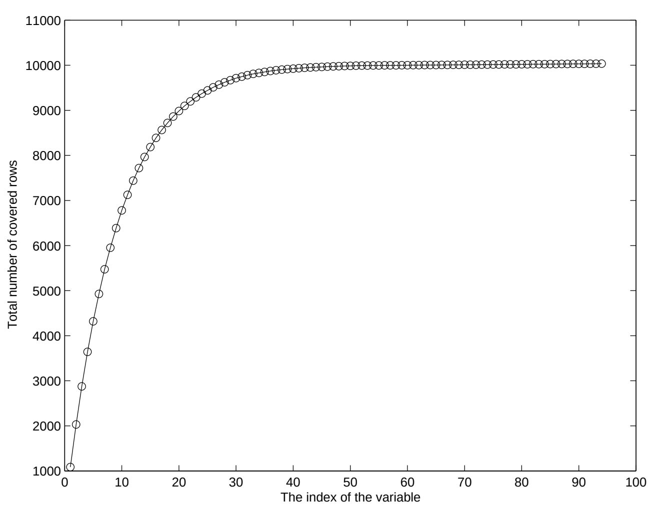
- Which functions are submodular?
- The concepts should not have interaction
- Variable selection?

Some experiments

10000 rows, 100 variables, each true with probability 0.1

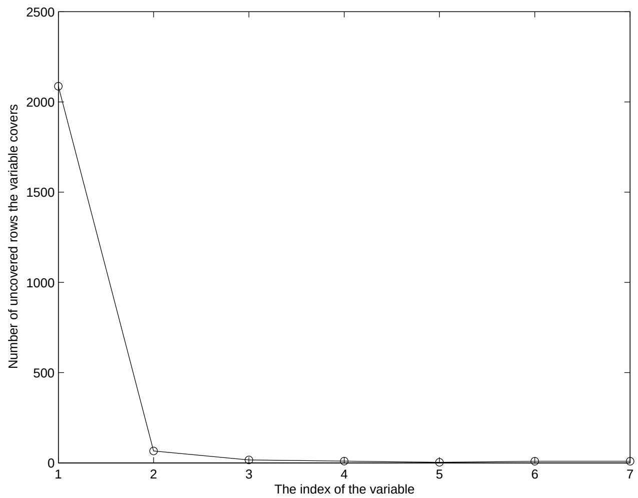


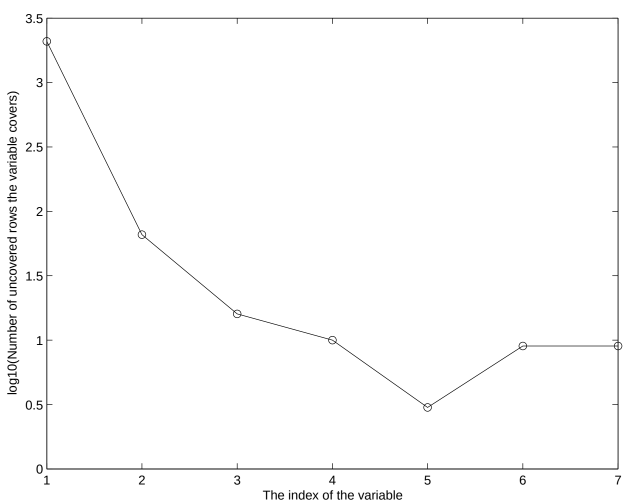

log10(Number of uncovered rows the variable covers)
The index of the variable

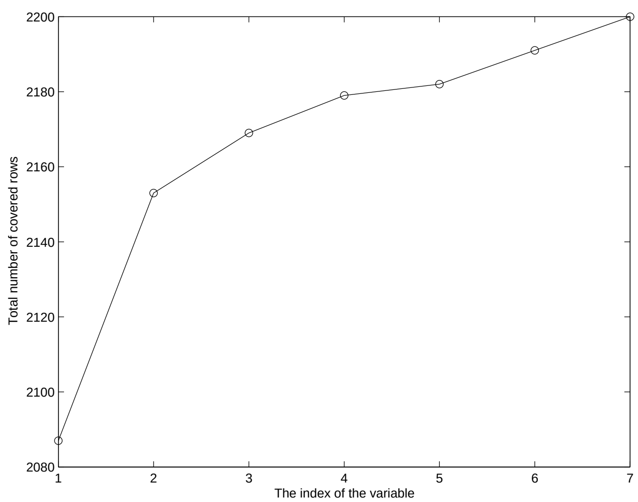

Total number of covered rows
The index of the variable

Other dataset

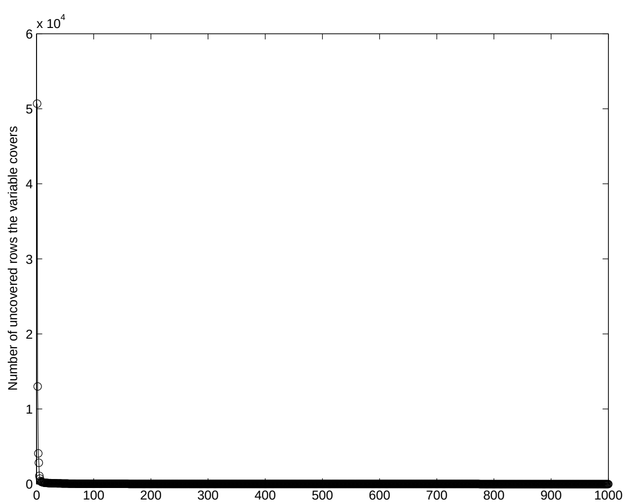
European mammals: 2183 rows, 124 variables (presence/absence in a grid)

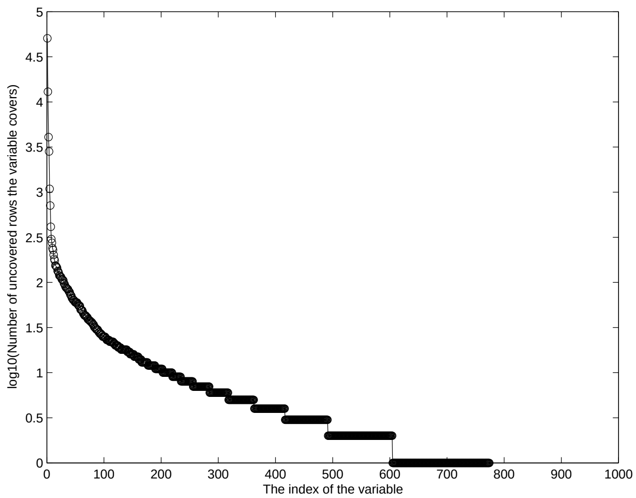


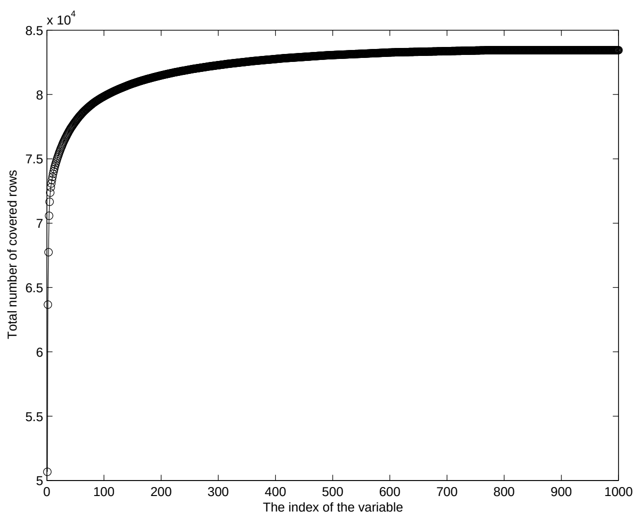




Retail data, 88162 rows, 1000 columns




log10(Number of uncovered rows the variable covers)
The index of the variable
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0
100
200
300
400
500
600
700
800
900
1000


x 10^4
Total number of covered rows
The index of the variable
5
5.5
6
6.5
7
7.5
8
8.5
0
100
200
300
400
500
600
700
800
900
1000