T-61.5040 Oppivat mallit ja menetelmät
T-61.5040 Learning Models and Methods
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Exercises 7, 2.3 2007

Problem 1. Assume you have two coins, 1 and 2, and you toss each one $n = 20$ times. The first coin gives $y_1 = 12$ heads and the second one $y_2 = 9$ heads. The probability of heads for coin 1 is θ_1, and for coin 2 it is θ_2.

i) Perform Bayesian inference on θ_1 and θ_2, assuming that the coins are independent and your prior for θ_i is $Beta(\theta_i|a, b)$.

ii) Repeat the same, assuming that $\theta = \theta_1 = \theta_2$

iii) The previous two parts either assumed that the coins have nothing to do with each other, or they are completely identical. Now assume that θ_i has a prior $Beta(\theta_i|a, b)$, and the variables a, b have exponential priors $Exp(a|1)$, $Exp(b|1)$. Write the posterior in the form

$$p(\theta_1, \theta_2, a, b|y_1, y_2) = p(\theta_1, \theta_2|a, b, y_1, y_2)p(a, b|y_1, y_2)$$

Hint:

$$Beta(\theta|a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$$

$$Bin(y|n, \theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

$$Exp(x|\lambda) = \lambda e^{-\lambda x}$$

Problem 2.

Assume that y is Normally distributed with mean θ and variance σ^2, both of which are unknown. Use the Jeffreys' prior separately for the unknown parameters, so that $p(\mu, \sigma^2) = p(\mu)p(\sigma^2)$.

i) Find the full posterior $p(\mu, \sigma^2|y)$

ii) Find the conditional posterior $p(\mu|\sigma^2, y)$

iii) Integrate μ out from $p(\mu, \sigma^2|y)$
Hint: write the integral as $C \int N(\mu|a, b)d\mu$, which equals C, since $N(\mu|a, b)$ is a probability distribution.

iv) Find the posterior $p(\mu|y)$
Hint: a Gamma integral is $\int_0^\infty z^k \exp(-z)dz$ and a substitution $z = (y-\mu)^2/2\sigma^2$ will give you such an integral.

Problem 3.

You have observed $x_1, \ldots, x_n$ from a Normal distribution $N(\theta_1, \sigma^2)$ and $y_1, \ldots, y_m$ from $N(\theta_2, \sigma^2)$. The means θ_1 and θ_2 have a common prior $N(\mu, \tau^2)$ where μ and τ are hyperparameters. Use priors $p(\mu) \propto 1$, $p(\sigma^2) \propto \sigma^{-2}$and $p(\tau^2) \propto \tau^{-1}$. Denote all data by D.

i) Find the distributions $p(\theta_i|\mu, \sigma, \tau, D)$, $i = 1, 2$.

ii) Find the distribution $p(\mu|\theta_1, \theta_2, \sigma, \tau, D)$

iii) Find the distribution $p(\sigma^2|\theta_1, \theta_2, \mu, \tau, D)$

iv) Find the distribution $p(\tau^2|\theta_1, \theta_2, \mu, \sigma, D)$

Hint: You will need one basic result not given in the lectures. An inverse-gamma distribution $IG(z|a, b)$ has the density $IG(z|a, b) \propto z^{-(a+1)} \exp(-b/z)$. If you have a Normal distribution $N(\mu, \sigma^2)$ where μ is known, and a prior for the variance given by $p(\sigma^2) = IG(\sigma^2|a, b)$, then using n observations v_i and writing $v = \frac{1}{n}\sum_i(v_i - \mu)^2$, the posterior for σ^2 is

$$p(\sigma^2|D) = IG\left(\sigma^2 \,\middle|\, \frac{n}{2} + a, \frac{1}{2}(2b + nv)\right).$$

Use this result in parts iii) and iv).

Problem 4.

(Demonstration.) Assume that you have data $y_1, \ldots, y_n$ where the likelihood is $p(y_i|\theta, \sigma_i^2) = N(y_i|\theta, \sigma_i^2)$. In other words, each sample is Normally distributed with mean θ and sample-dependent variance σ_i^2. Assume a constant prior $p(\theta|\sigma_i^2)$ and a prior $p(\sigma_i^2) \propto \sigma_i^{-7} \exp(-2\sigma_i^{-2})$. Compute the posterior distribution of θ. Suppose the data is $0, 0, 0, 0, 0, 4$: compare the values $\theta = 0$ and $\theta = 1$ using this posterior, and using directly the likelihood with fixed variance.