Answers 6

3. Consider the following 2D data set in (x_1, x_2) coordinates

$$\{(-1,3), (-1,0), (-1,-3), (1,3), (1,0), (1,-3)\}. \tag{1}$$

Whiten the data and calculate the variance and kurtosis in the direction of the x_1 and x_2 axis. Find the direction of the largest PCA and ICA component. You can use Matlab with the FastICA package available from Infolab to find the solutions.

Variance is $E\{x\}$ and kurtosis is $E\{x^4\} - 3(E\{x^2\})^2$. Ignoring the question whether the correct denominator is N or $N-1$, the values are:

	mean	var	kurtosis	kurtosis when normalized to unit variance
1	0	1	-2	-2
2	0	6	-54	-1.775

PCA: Eigen vectors and eigen values

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \tag{2}$$

ICA: Solution weights

$$W = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix} \tag{3}$$

6. You have three points in 2D space.

$$\begin{aligned} A &= (1,2) \\ B &= (3,4) \\ C &= (3,0) \end{aligned}$$

Find a 1D MDS representation for the points using a MDS model of your choice. You should write the solution equations, but the numerical solution can be solved using numerical methods. HINT: You can fix the position of one of the points and solve for the positions of the other two.

Cost function:

$$E = \sum_i \sum_j \left(d_{ij} - \hat{d}_{ij}\right)^2 = \sum_{ij} \left(d_{ij} - \|x_i - x_j\|\right)^2 \tag{4}$$

We need the following for the gradient of E:

$$\frac{\partial \|x_i - x_j\|}{\partial x_k^{(l)}} = \frac{\partial \sum_t \sqrt{(x_k^{(t)} - x_j^{(t)})^2}}{\partial x_k^{(l)}} = \frac{x_k^{(l)} - x_j^{(l)}}{\|x_k - x_j\|} \tag{5}$$

The gradient of E:

$$\begin{aligned} \frac{\partial E}{\partial x_k^{(l)}} &= \sum_{ij} 2\left(d_{ij} - \|x_i - x_j\|\right)\left(-\frac{\partial \|x_i - x_j\|}{\partial x_k^{(l)}}\right) && (6) \\ &= 2\sum_j \left[-2d_{kj}\frac{x_k^{(l)} - x_j^{(l)}}{\|x_k - x_j\|} + 2(x_k^{(l)} - x_j^{(l)})\right] && (7) \\ &= 4\sum_j \left(1 - \frac{d_{kj}}{\|x_k - x_j\|}\right)\left(x_k^{(l)} - x_j^{(l)}\right) && (8) \end{aligned}$$

For 1-diemensional case this becomes:

$$\frac{\partial E}{\partial x_k} = 4\sum_j \left[1 - \frac{d_{kj}}{\|x_k - x_j\|}(x_k - x_j)\right] \tag{9}$$

We select the point C to be positioned at zero, the distance matrix is

	A	B	C
A	0	8	8
B	8	0	16
C	8	16	0

(10)

and make the asumption that $0 \leq x_A \leq x_b$. Now set the derivatives to zero

$$\begin{aligned}
\frac{\partial E}{\partial x_A} &= (1 - \frac{d_{AB}}{x_B - x_A})(x_A - X_B) + x_A - d_{AC} = 0 \qquad (11) \\
\frac{\partial E}{\partial x_B} &= (1 - \frac{d_{BA}}{x_B - x_A})(x_B - X_A) + x_B - d_{BC} = 0 \qquad (12)
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
x_A - x_B + d_{AB} + x_A - d_{AC} &= 0 \qquad (13) \\
x_B - x_A + d_{AB} + x_B - d_{BC} &= 0 \qquad (14) \\
& \qquad (15)
\end{aligned}$$

Some substitutions and we get

$$\begin{aligned}
x_B &= 2x_A + d_{AB} - d_{AC} \qquad (16) \\
x_A &= \frac{2d_{AC} + d_{BC} - d_{AB}}{3} \qquad (17)
\end{aligned}$$

Substitute actual values

$$\begin{aligned}
x_C &= 0 \qquad (18) \\
x_A &= \frac{2}{3}(2 + \sqrt{2}) \approx 2.28 \qquad (19) \\
x_B &= \frac{2}{3}(4 + \sqrt{8}) \approx 4.55 \qquad (20)
\end{aligned}$$