T-61.3010 Digital Signal Processing and Filtering

T-61.3010 Digitaalinen signaalinkäsittely ja suodatus

(B) Exercise material for spring 2006 by professor Olli Simula and assistant Jukka Parviainen. Corrections and comments to `t613010@cis.hut.fi`, thank you!

This material is intended for "paper sessions" on Tuesdays 12-14 L, on Wednesdays 10-12 G, and on Thursdays 14-16 G, in spring 2006.

The course follows the book "Digital Signal Processing" by Sanjit K. Mitra. There are three different editions available, 3rd being the newest. Notation (*Mitra 2Ed Sec. 5.2 / 3Ed Sec. 4.2*) refers to the section 5.2 in the 2nd Edition (yellow cover) of Mitra's Book and to the section 4.2 in the 3rd Edition (blue, antenna). There is a brief correspondence table of three editions and errata lists in the course web pages `http://www.cis.hut.fi/Opinnot/T-61.3010/`. Course lecture slides by Olli Simula follow the second edition of Mitra's book.

This copy belongs to:

Contens

Description of Example Problems

#	Subject	M 2Ed	M 3Ed
Math Background 1-11			
1	complex numbers, Carthesian and polar coordinate systems, Euler's formula		
2	Euler's formula, cosine and sine		
3	complex numbers, graphical notation		
4	complex-valued function		
5	cosine function, amplitude, frequency, phase		
6	logarithm, decibels, sinc, modulo, binary number representation		
7	roots of a polynomial		
8	complex-valued function, roots of polynomial		
9	partial fraction expansion / decomposition		
10	sum of geometric series		
11	integral transforms		*Sec. 5.1*
Discrete-time Signals and Systems in Time Domain 12-25		*Sec. 2*	*Sec. 2*
12	signals and sequences, unit impulse and unit step functions ($\delta[n]$, $\mu[n]$)		
13	periodic signals		
14	flow / block diagram of a discrete-time system		
15	recognition of LTI systems, causal LTI systems, filter order		
16	properties of LTI systems: linear, time-invariant, causal, stable		
17	impulse response $h[n]$		
18	step response $s[n]$		
19	linear convolution $y(t) = x_1(t) \circledast x_2(t)$ of continuous-time signals		
20	linear convolution $y[n] = h[n] \circledast x[n]$ of discrete-time signals		
21	convolution as products of polynomials		
22	deconvolution		
23	parallel and cascade (series) LTI systems		
24	matched filter		
25	auto- and cross-correlation		
Discrete-time Signals and Systems in Frequency Domain 26-30		*Sec. 3*	*Sec. 3, 5*
26	spectrum, DTFT, DFT		
27	spectrum, DTFT		
28	DTFT, computation from definition		
29	DFT		
30	circular convolution		
System Analysis in Frequency Domain 31-35		*Sec. 4*	*Sec. 3.8-, 6, 7*
31	analysis of LTI system in frequency domain		
32	amplitude response grafically from pole-zero-plot		
33	analysis of LTI system in frequency domain		
34	transfer function, z-transform, region of convergence (ROC)		

continued on next page

continued from previous page			
#	Subject	M 2Ed	M 3Ed
35	filter types: allpass, zero-phase, linear-phase, minimum-phase, maximum-phase		
Sampling and Aliasing 36-40		*Sec. 5*	*Sec. 4*
36	impulse train and Fourier-series		
37	sampling in frequency domain		
38	sampling in frequency domain		
39	aliasing		
40	anti-aliasing filter		
System Structures 41-45		*Sec. 6*	*Sec. 8*
41	LTI subsystems		
42	polyphase structure		
43	canonic structure		
44	direct form (DF) structures		
45	direct form, transpose		
Digital Filter Design 46-51		*Sec. 7*	*Sec. 9, 10*
46	filter specifications		
47	analog filter approximations		
48	scaling factor		
49	bilinear transform and impulse-invariant method in digital filter design		
50	FIR-window method in digital filter design		
51	computational issues on IIR / FIR filters		
Implementation Issues 52-56		*Sec. 8, 9*	*Sec. 11, 12*
52	FFT		
53	fixed-point binary number representations		
54	quantization, error densities		
55	quantization noise		
56	error-feedback structure		
Multirate Systems 57-60		*Sec. 10*	*Sec. 13, 14*
57	up- and downsampling in time- and frequency domain		
58	multirate system analysis		
59	linearity of up- and downsampling systems		
60	filter bank		

T-61.3010 Digital Signal Processing and Filtering
Example problems for spring 2006.
Solutions start from Page 23.

Problems

Math Background 1-11

1. Complex numbers in Carthesian (rectangular) coordinates $z = x + yj$ (or i) and polar coordinates $z = r \cdot e^{j\theta}$. The complex conjugate z^* is $z^* = x - yj = r \cdot e^{-j\theta}$. Euler's formula $e^{j\omega} = \cos(\omega) + j\ \sin(\omega)$.

   a) Express $z = 2e^{-j\pi}$ in rectangular coordinates.
   b) Express $z = -1 + 2j$ in polar coordinates.
   c) Which (two) angles satisfy $\sin(\omega) = 0.5$?
   d) What are $z + z^*$, $|z + z^*|$? and $\angle(z + z^*)$? What are zz^*, $|zz^*|$? and $\angle zz^*$?

2. The important Euler's formula is $e^{j\theta} = \cos(\theta) + j \cdot \sin(\theta)$. Cosine is even function $f(x) = f(-x)$ and sine is odd function $f(x) = -f(-x)$.

   a) Express with cosines and sines: $e^{j\theta} + e^{j(-\theta)}$.
   b) Express with cosines and sines: $e^{j\theta} - e^{j(-\theta)}$.
   c) Express with cosines and sines: $e^{j\pi/8} \cdot e^{j\theta} - e^{j(-\pi/8)} \cdot e^{j(-\theta)}$.

3. Consider the following three complex numbers

$$\begin{aligned} z_1 &= 3 + 2j \\ z_2 &= -2 + 4j \\ z_3 &= -1 - 5j \end{aligned}$$

   a) Draw the vectors z_1, z_2, and z_3 separately in complex plane.
   b) Draw and compute the sum $z_1 + z_2 + z_3$.
   c) Draw and compute the weighted sum $z_1 - 2z_2 + 3z_3$.
   d) Draw and compute the product $z_1 \cdot z_2 \cdot z_3$.
   e) Compute and reduce the division z_1/z_2.

4. Examine a complex-valued function

$$H(\omega) = 2 - e^{-j\omega}$$

   where $\omega \in [0 \ldots \pi] \in \mathbb{R}$.

   a) Compute values of Table 1 with a calculator. Euler: $e^{j\omega} = \cos(\omega) + j\ \sin(\omega)$.
   b) Draw the values at $\omega = \{0,\ \pi/4,\ \ldots,\ \pi\}$ into complex plane $(x,\ y)$. Interpolate smoothly between the points.
   c) Sketch $|H(\omega)|$ as a function of ω. Interpolate.
   d) Skecth $\angle H(\omega)$ as a function of ω. Interpolate.

5. A cosine signal can be represented using its angular frequency Ω or frequency f, amplitude A and phase θ:

$$x(t) = A\,\cos(\Omega t + \theta) = A\,\cos(2\pi f t + \theta)$$

   a) Estimate $A,\ f,\ \theta$ for the cosine $x_1(t)$ in Figure 1(a).
   b) Sketch a cosine $x_2(t)$, with $A = 2$, angular frequency 47 rad/s and angle $-\pi/2$.
   c) Express $x_2(t)$ in (b) using exponential functions.

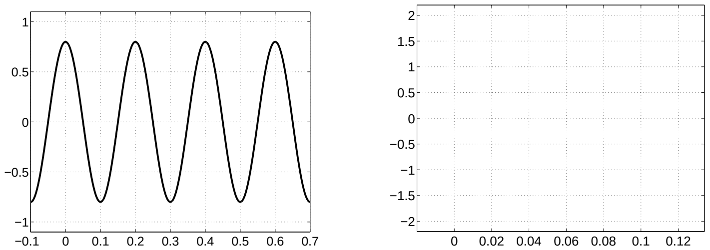


Figure 1: Cosine $x_1(t)$ (left) and $x_2(t)$ (right) in Problem 5.

6. Some elementary functions and notations.

   a) Compute with a calculator: $\log_8 7$.
   b) The power of signal is attenuated from 10 to 0.01. How much is the attenuation in decibels?
   c) Sketch the curve $p(x) = \sum_{k=-N}^{+N} kx$ for various N.
   d) Sinc-function is useful in the signal processing. It is defined $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. Also it is known that $\sin(x)/x \to 1$, when $x \to 0$, and with sinc-function $\mathrm{sinc}(0) = 1$. Consider $h(n) = \sin(0.75\pi n)/(\pi n)$. What is $h(0)$?
   e) Modulo-N operation for number x is written here as $< x >_N$. What is $< -4 >_3$?
   f) What is the binary number $(1001011)_2$ as a decimal number?

7. Roots of a polynomial $p(x)$ can be found from $p(x) = 0$. Nth root of $z = r\,e^{j(\theta + 2\pi k)}$ is $\sqrt[N]{z} = |\sqrt[N]{r}| \cdot e^{j(2\pi\,k/N + \theta/N)}$, where $k = 0 \ldots N-1$.

   a) Compute roots of $H(z) = z^2 + 2z + 2$.
   b) Compute roots of $H(z) = 1 + 16z^{-4}$.
   c) Compute long division $(4z^4 - 8z^3 + 3z^2 - 4z + 6)/(2z - 3)$.

ω	$x = \mathrm{Real}(H(\omega))$	$y = \mathrm{Imag}(H(\omega))$	$r = \lvert H(\omega)\rvert$	$\theta = \angle H(\omega)$
0				
$\pi/4$				
$\pi/2$				
$3\pi/4$				
π				

Table 1: Problem 4: values of a complex-valued function in rectangular $(x,\ y)$ and polar $(r,\ \theta)$ coordinates.

8. Examine a complex-valued function ($z \in \mathbb{C}$)

$$H(z) = \frac{1 + 0.5z^{-1} + 0.06z^{-2}}{1 - 1.4z^{-1} + 0.48z^{-2}}$$

   a) Multiply both sides by z^2.
   b) Solve $z^2 + 0.5z + 0.06 = 0$.
   c) Solve $z^2 - 1.4z + 0.48 = 0$.
   d) $H(z)$ can be written with five values complex values K, z_1, z_2, p_1, and p_2

$$H(z) = K \cdot \frac{(z - z_1)\cdot(z - z_2)}{(z - p_1)\cdot(z - p_2)}$$

   What are the five values?
   e) What are the coefficients of $H(z)$. What are the roots of $H(z)$? What is the order of the numerator polynomial of $H(z)$? What is the order of the denominator polynomial of $H(z)$?

9. Partial fraction expansion (osamurtohajotelma, osamurtokehitelmä) is used to divide a high-order rational expression into a sum of low-order rational expressions. For example, $1/(x^2 + 3x + 2) = 1/(x + 1) - 1/(x + 2)$.

   Decomposition is quite trivial if there are not multiple roots neither is the order of numerator polynomial as big or bigger as the order of the denominator polynomial. For more complicated cases, see Mitra Sec. 3.9., or any other math reference.

   a) Decompose $f(x) = 1/(x^2 + 1)$ into sum of first-order expressions.
   b) Decompose $H(z) = (0.4 - 0.2z^{-1})/(1 - 0.1z^{-1} - 0.06z^{-2})$ into sum of first-order expressions.

10. When the ratio q in geometric series is $|q| < 1$, the sum of series converges to $\sum_{k=0}^{\infty} q^k = 1/(1-q)$, and correspondingly $\sum_{k=0}^{N} q^k = (1 - q^{N+1})/(1 - q)$.

   Other known series are $\{1/n\}$ and $\{1/n^2\}$. Notice that the former does not converge, while the latter does.

   a) What is sum of series $S = \sum_{k=0}^{\infty} (0.5)^k$.
   b) $S = \sum_{k=10}^{\infty} (-0.6)^{k-2}$.
   c) $S = \sum_{k=2}^{\infty} (0.8^{k-2} \cdot e^{-j\omega k})$.

11. Integral transforms, like Fourier-transforms, play an important role in signal processing.

   a) List all integral transforms that are used in previous signal processing courses.
   b) Compute the integral $X(\Omega) = \int_0^4 e^{-j\Omega t}\mathrm{d}t$.

Discrete-time Signals and Systems in Time Domain 12-25

12. The unit impulse function $\delta[n]$ and the unit step function $\mu[n]$ (or $u[n]$) are defined

$$\delta[n] = \begin{cases} 1, & \text{when } n = 0 \\ 0, & \text{when } n \neq 0 \end{cases} \qquad \mu[n] = \begin{cases} 1, & \text{when } n \geq 0 \\ 0, & \text{when } n < 0 \end{cases}$$

   Sketch the following sequences around the origo

a) $x_1[n] = \sin(0.1\pi n)$

b) $x_2[n] = \sin(2\pi n)$

c) $x_3[n] = \delta[n-1] + \delta[n] + 2\delta[n+1]$

d) $x_4[n] = \delta[-1] + \delta[0] + 2\delta[1]$

e) $x_5[n] = \mu[n] - \mu[n-4]$

f) $x_6[n] = x_3[-n+1]$

13. Continuous-time signal $x(t)$ is periodic, if there exists period $T \in \mathbb{R}$, for which $x(t) = x(t+T)$, $\forall t$. Discrete-time signal (sequence) $x[n]$ is periodic, if $\exists N \in \mathbb{Z}$, for which $x[n] = x[n+N]$, $\forall n \in \mathbb{Z}$. The fundamental period T_0 (or N_0) is the smallest period bigger than 0.

    Which of the following signals are periodic? Define the length of the fundamental period for periodic signals.

    a) $x(t) = 3\cos(\frac{8\pi}{31}t)$

    b) $x[n] = 3\cos(\frac{8\pi}{31}n)$

    c) $x(t) = \cos(\frac{\pi}{8}t^2)$

    d) $x[n] = 2\cos(\frac{\pi}{6}n - \pi/8) + \sin(\frac{\pi}{8}n)$

    e) $x[n] = \{\ldots, \underline{2}, 0, 1, 2, 0, 1, 2, 0, 1, \ldots\}$

    f) $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-4k] + \delta[n-4k-1]$

14. There are some basic operations on sequences (signals) in discrete-time systems (x refers to input to the system / operation, y output) shown also in Figure 2.

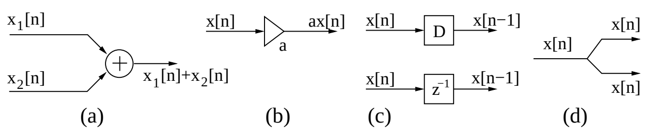


Figure 2: Problem 14: Basic operations in discrete-time systems, (a) sum of sequencies, (b) amplification by constant, (c) unit delay (D, T, or z^{-1}), and (d) branch / pick-off node.

Express the input-output relations of the discrete-time systems in Figure 3.

15. Look at the flow (block) diagrams in Figure 4.

    a) What does LTI mean? In what ways can the system be proved (Problem 16) or shown to be LTI?

    b) Which systems are linear and time-invariant (LTI) without any computation?

    c) Which systems have feedback?

    d) Which LTI systems are FIR and which are IIR?

16. For each the following discrete-time systems, determine whether or not the system is (1) linear, (2) causal, (3) stable, and (4) shift-invariant. The sequences $x[n]$ and $y[n]$ are the input and output sequences of the system.

    a) $y[n] = x^3[n]$,

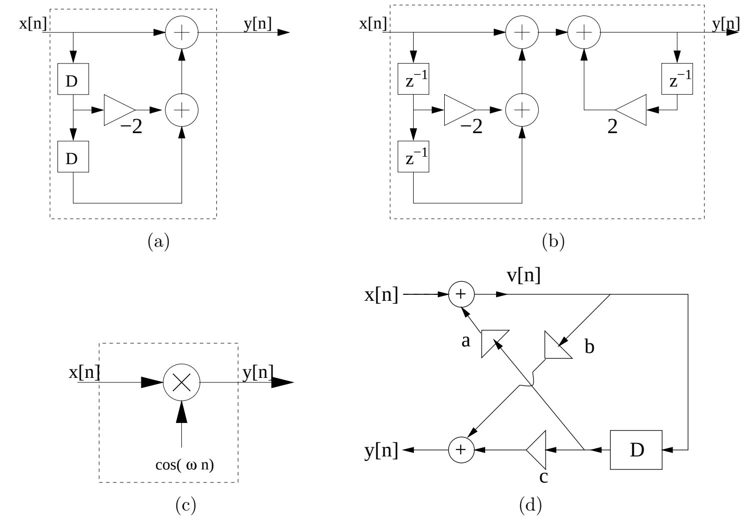


Figure 3: Discrete-time systems for Problems 14, 17, and 18.

b) $y[n] = \gamma + \sum_{l=-2}^{2} x[n-l]$, γ is a nonzero constant,

c) $y[n] = \alpha x[-n]$, α is a nonzero constant.

17. Impulse response $h[n]$ is the response of the system to the input $\delta[n]$.

    a) What is the impulse response of the system in Figure 3(a)? What is the connection to the difference equation? Is this LTI system stable/causal?

    b) What are the first five values of impulse response of the system in Figure 3(b)? Hint: Fetch the input $\delta[n]$ and read what comes out. Is it possible to say something about stability or causality of the system?

    c) What are the first five values of impulse response of the system in Figure 3(d)?

18. Step response $s[n]$ is the response of the system to the input $\mu[n]$. What are the step responses of systems in Figures 3(a) and (b)?

19. Compute the convolution of two signals $x_1(t)$ and $x_2(t)$ in both cases (a) and (b) in Figure 5. The arrows in (b) are impulses $\delta(t)$.

20. Linear convolution of two sequences is defined

$$y[n] = h[n] \circledast x[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

    a) Compute $x[n] \circledast h[n]$, when
    $x[n] = \delta[n] + \delta[n-1]$, and $h[n] = \delta[n] + \delta[n-1]$.
    What is the length of the convolution result?

    b) Compute $x_1[n] \circledast x_2[n]$, when
    $x_1[n] = \delta[n] + 5\delta[n-1]$, and $x_2[n] = -\delta[n-1] + 2\delta[n-2] - \delta[n-3] - 5\delta[n-4]$.
    What is the length of the convolution result? Where does the output sequence start?

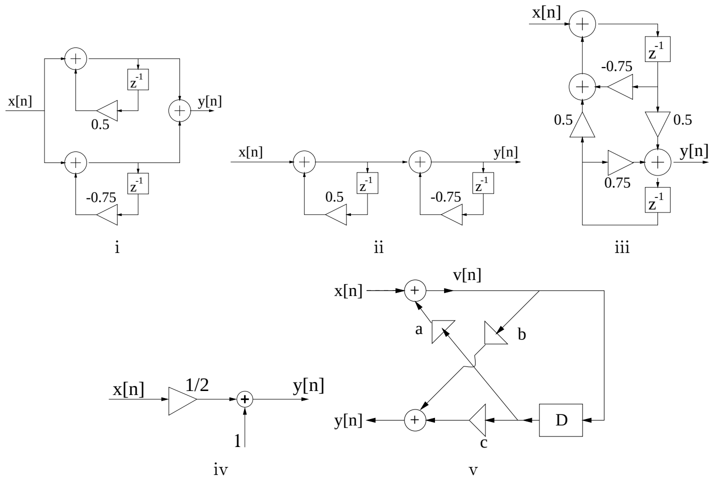


Figure 4: Flow diagrams of Problem 15.

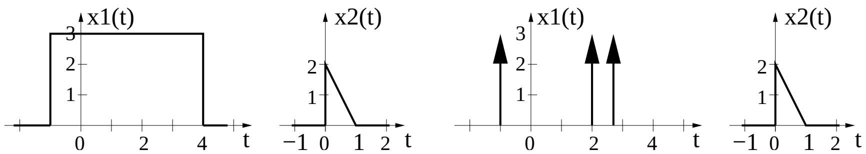


Figure 5: Problem 19: signals $x_1(t)$ and $x_2(t)$ to be convolved, left: (a), right: (b).

c) Compute $h[n] \circledast x[n]$, when
$h[n] = 0.5^n \mu[n]$, and $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$.
What is the length of the convolution result?

21. Consider a LTI-system with impulse response $h[n] = \delta[n-1] - \delta[n-2]$ and input sequence $x[n] = 2\delta[n] + 3\delta[n-2]$.

   a) What is the length of convolution of $h[n]$ and $x[n]$ (without computing convolution itself)? Which index n is the first one having a non-zero item?

   b) Compute convolution $y[n] = h[n] \circledast x[n]$

   c) Consider polynomials $S(x) = 2 + 3x^2$ and $T(x) = x - x^2$. Compute the product $U(x) = S(x) \cdot T(x)$

   d) Check the result by computing the polynomial division $T(x) = U(x)/S(x)$.

22. The impulse response $h_1[n]$ of a LTI system is known to be $h_1[n] = \mu[n] - \mu[n-2]$. It is connected in cascade (series) with another LTI system h_2 as shown in Figure 6.

   Compute the impulse response $h_2[n]$, when it is known that the impulse response $h[n]$ of the whole system is shown in Table 2 below.

23. LTI systems are commutative, distributive and associative. Determine the expression for the impulse response of each of the LTI systems shown in Figure 7.

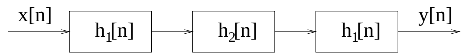


Figure 6: The cascade system of Problem 22.

n	< 0	0	1	2	3	4	> 4
h[n]	0	1	5	9	7	2	0

Table 2: Impulse response of the cascade system in Problem 22.

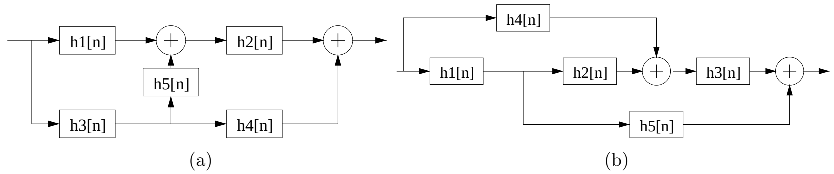


Figure 7: LTI systems in Problem 23.

24. The impulse response of a digital matched filter, $h[n]$, is the time-reversed replica of the signal to be detected. The time-shift is needed in order to get a causal filter.

   The (binary) signal to be detected is given by $s[n] = \{1, 1, 1, -1, -1, 1, -1\}$. Consider an input sequence $x[n]$ which is a periodic sequence repeating $s[n]$. Determine $h[n]$ and the result of filtering $y[n] = h[n] \circledast x[n]$.

25. Cross-correlation sequence $r_{xy}[l]$ of two sequences and autocorrelation sequence $r_{xx}[l]$ with lag $l = 0, \pm 1, \pm 2, \ldots$ are defined

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l] \qquad r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

   Determine the autocorrelation sequence of the sequence

$$x_1[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

   and show that it is an even sequence. What is the location of the maximum value of the autocorrelation sequence?

Discrete-time Signals and Systems in Frequency Domain 26-30

26. Sketch the following signals in time-domain and their (amplitude) spectra in frequency-domain.

   a) $x_1(t) = \cos(2\pi\, 500\, t)$

   b) $x_2(t) = 4\cos(2\pi\, 200\, t) + 2\sin(2\pi\, 300\, t)$

   c) $x_3(t) = e^{-j(2\pi\, 250 t)} + e^{j(2\pi\, 250 t)}$

   d) $x_4(t) = x_1(t) + x_2(t) + x_3(t)$

27. Suppose that a real sequence $x[n]$ and its discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ are known. The sampling frequency is f_s. At angular frequency $\omega_c = \pi/4$: $X(e^{j(\pi/4)}) = 3 + 4j$. Determine

    a) $|X(e^{j(\pi/4)})|$ b) $\angle X(e^{j(\pi/4)})$
    c) $X(e^{j(-\pi/4)})$ d) $X(e^{j(\pi/4+2\pi)})$
    e) If $f_s = 4000$ Hz, what is f_c

28. Compute DTFT for each of the following sequences using the definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

    a) $x_1[n] = \delta[n-2]$
    b) $x_2[n] = 0.5^n \mu[n]$
    c) $x_3[n] = a[n] \cdot \cos(\frac{\pi}{5}n)$

29. The exponent term in DFT/IDFT is commonly written $W_N = e^{-j2\pi/N}$.

    a) Compute and draw in complex plane values of W_3^k
    b) Compute 3-DFT for the sequence $x[n] = \{1,\ 3,\ 2\}$.

30. Let $g[n]$ and $h[n]$ be two finite-length sequences given below:

$$g[n] = \begin{cases} 5, & \text{for } n = 0, \\ 2, & \text{for } n = 1, \\ 4, & \text{for } n = 2 \end{cases} \qquad h[n] = \begin{cases} -3, & \text{for } n = 0, \\ 4, & \text{for } n = 1, \\ 0, & \text{for } n = 2, \\ 2, & \text{for } n = 3 \end{cases}$$

    a) Determine the linear convolution $y_L[n] = g[n] \circledast h[n]$.
    b) Extend $g[n]$ to length-4 sequence $g_e[n]$ by zero-padding and compute the circular convolution $y_C[n] = g_e[n] \textcircled{4} h[n]$.
    c) Extend both sequences to length-6 sequences by zero-padding and compute the circular convolution $y_C[n] = g_e[n] \textcircled{6} h_e[n]$. Show that now $y_C[n] = y_L[n]$!

System Analysis in Frequency Domain 31-35

31. Consider a LTI system depicted in Figure 8 with registers having initial values of zero and the input sequence $x[n] = (-0.8)^n \mu[n]$.

    a) What is the difference equation of the system?
    b) Compute $X(z)$ using the definition of z-transform or consult the z-transform table.
    c) Apply z-transform to the difference equation. What is the transfer function $H(z) = Y(z)/X(z)$? Where are the constant multipliers of the system seen in Figure 8 in difference equation and in transfer function? Hint: the z-transform of $K\,w[n-n_0]$ is $K\,z^{-n_0}\,W(z)$.
    d) Now it is possible to compute the output $y[n]$ without convolution in time-domain using the convolution theorem

$$y[n] = h[n] \circledast x[n] \quad \leftrightarrow \quad Y(z) = H(z) \cdot X(z)$$

    Write down the equation for $Y(z)$, use partial fraction expansion in order to achieve rational polynomials of first order, and then use the inverse z-transform (equation in (b)).

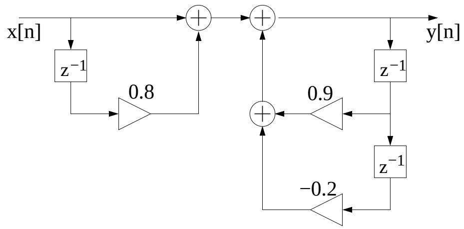


Figure 8: LTI system of Problem 31.

32. Consider the pole-zero plots in Figure 9.

    a) What is the order of each transfer function?
    b) Are they FIR or IIR?
    c) Sketch the amplitude response for each filter.
    d) What could be the transfer function of each filter?

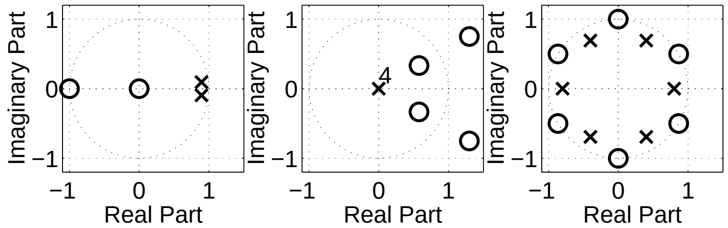


Figure 9: Pole-zero plots of LTI systems in Problem 32.

33. Consider the filter described in Figure 10.

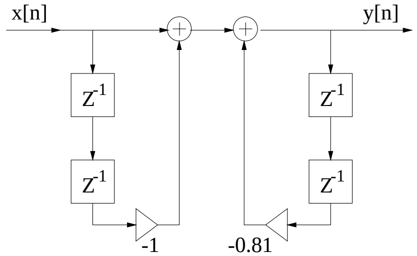


Figure 10: LTI system of Problem 33.

    a) Derive the difference equation of the system.
    b) Calculate the transfer function $H(z)$.
    c) Calculate the zeros and poles of $H(z)$. Sketch the pole-zero plot. Is the system stable and/or causal?

d) If the region of convergence (ROC) of $H(z)$ includes the unit circle, it is possible to derive frequency response $H(e^{j\omega})$ by applying $z = e^{j\omega}$. Do this!

e) Sketch the magnitude (amplitude) response $|H(e^{j\omega})|$ roughly. Which frequency gives the maximum value of $|H(e^{j\omega})|$? (If you want to calculate magnitude response explicitely, calculate $|H(e^{j\omega})|^2 = H(e^{j\omega})H(e^{-j\omega})$ and use Euler's formula.)

f) Compute the equation for the impulse response $h[n]$ using partial fraction expansion and inverse z-transform.

34. The transfer function of a filter is

$$H(z) = \frac{1 - z^{-1}}{1 - 2z^{-1} + 0.75z^{-2}}$$

a) Compute the zeros and poles of $H(z)$.

b) What are the three different regions of convergence (ROC)?

c) Determine the ROC and the impulse response $h[n]$ so that the filter is causal.

d) Determine the ROC and the impulse response $h[n]$ so that the filter is stable.

35. Examine the following five filters and connect them at least to one of the following categories (a) zero-phase, (b) linear-phase, (c) allpass, (d) minimum-phase, (e) maximum-phase.

$$\begin{aligned}
h_1[n] &= -\delta[n+1] + 2\delta[n] - \delta[n-1] \\
H_2(z) &= \frac{1 + 3z^{-1} + 2.5z^{-2}}{1 - 0.5z^{-1}} \\
y_3[n] &= 0.5y_3[n-1] + x[n] + 1.2x[n-1] + 0.4x[n-2] \\
H_4(z) &= \frac{0.2 - 0.5z^{-1} + z^{-2}}{1 - 0.5z^{-1} + 0.2z^{-2}} \\
H_5(e^{j\omega}) &= -1 + 2e^{-j\omega} - e^{-2j\omega}
\end{aligned}$$

Sampling and Aliasing 36-40

36. Show that the periodic impulse train $p(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

can be expressed as a Fourier series

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j(2\pi/T)kt} = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j\Omega_T kt},$$

where $\Omega_T = 2\pi/T$ is the sampling angular frequency.

37. Impulse train in Problem 36 can be also expressed as a Fourier transform

$$P(j\Omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\,\Omega_s)$$

Sampling can be modelled as multiplication in time domain $x[n] = x_p(t) = x(t)p(t)$. What is $X_p(j\Omega)$ for an arbitrary input spectrum $X(j\Omega)$?

Hints: Fourier transform of a periodic signal (Fourier series)

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0)$$

Multiplication of signals in time domain corresponds to convolution of transforms in frequency domain:

$$x_1(t)\cdot x_2(t) \quad\leftrightarrow\quad \frac{1}{2\pi}\left[X_1(j\Omega) \circledast X_2(j\Omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\theta)\cdot X_2(j(\Omega-\theta))\mathrm{d}\theta$$

38. Suppose that a continuous-time signal $x(t)$ and its spectrum $|X(j\Omega)|$ in Figure 11 are known.

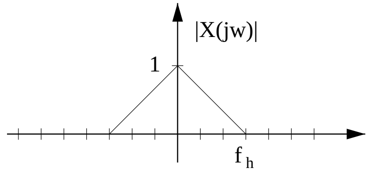


Figure 11: Spectrum $X(j\Omega)$ in Problem 38.

The highest frequency component in the signal is f_h. The signal is sampled with frequency f_s, i.e. the interval between samples is $T_s = 1/f_s$: $x[n] = x(nT_s)$. Sketch the spectrum $|X(e^{j\omega})|$ of the discrete-time signal, when

a) $f_h = 0.25\ f_s$

b) $f_h = 0.5\ f_s$

c) $f_h = 0.75\ f_s$

39. Consider a continuous-time signal

$$\tilde{x}(t) = \begin{cases} \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where f_1=100 Hz, f_2=300 Hz and f_3=750 Hz. The signal is sampled using frequency f_s. Thus, a discrete signal $x[n] = \tilde{x}(nT_s) = \tilde{x}(n/f_s)$ is obtained.

Sketch the magnitude of the Fourier spectrum of $x[n]$, the sampled signal, when f_s equals to (i) 1600 Hz (ii) 800 Hz (iii) 400 Hz.

Use an ideal reconstruction lowpass filter whose cutoff frequency is $f_s/2$ for each case. What frequency components can be found in reconstructed analog signal $x_r(t)$?

40. Suppose that there is an analog signal which will be sampled with 8 kHz. The interesting band is $0 \ldots 2$ kHz. Sketch specifications for an anti-aliasing filter. Determine the order of the filter when using Butterworth approximation and minimum stopband attenuation is 50 dB. The variables in Table 3: Ω_p is the passband edge frequency (interesting part), Ω_T is the sampling frequency, and Ω_0 is the frequency after which the aliasing components are small enough.

$\Omega_0 =$	$2\Omega_p$	$3\Omega_p$	$4\Omega_p$
Attenuation (dB)	$6.02N$	$9.54N$	$12.04N$
$\Omega_T =$	$3\Omega_p$	$4\Omega_p$	$5\Omega_p$

Table 3: Approximate minimum stopband attenuation of a Butterworth lowpass filter. Mitra Table 5.1, page 336. See the text in Problem 40 for details.

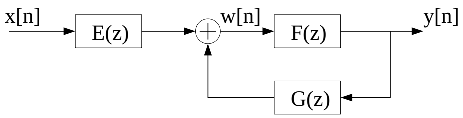


Figure 12: System in Problem 41.

System Structures 41-45

41. Derive the transfer function of the feedback system shown in Figure 12.

42. Develop a polyphase realization of a length-9 FIR transfer function given by

$$H(z) = \sum_{n=0}^{8} h[n]z^{-n}$$

with (a) 2 branches and (b) 4 branches.

43. Analyze the digital filter structure shown in Figure 13 and determine its transfer function $H(z) = Y(z)/X(z)$.

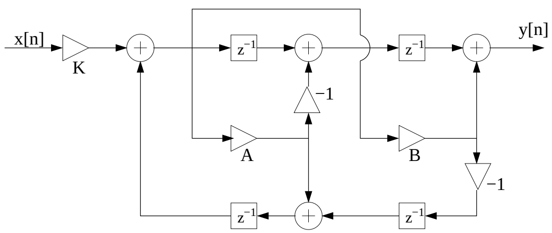


Figure 13: The flow diagram of the system in Problem 43.

.

a) Is the system LTI?
b) Is the structure canonic to delays?
c) Compute $H(z)H(z^{-1})$ (the squared amplitude response). What is the type of this filter (lowpass/highpass/bandpass/bandstop/allpass)?

44. The filter in Figure 14 is in canonic direct form II (DF II). Draw it in DF I. What is the transfer function $H(z)$?

45. Develop a canonic direct form realization of the transfer function

$$H(z) = \frac{2 + 4z^{-1} - 7z^{-2} + 3z^{-5}}{1 + 2z^{-1} + 5z^{-3}}$$

and then determine its transpose configuration.

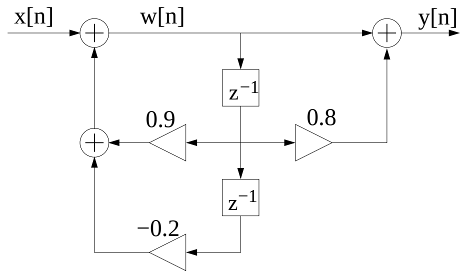


Figure 14: The block diagram of direct form II of Problem 44.

Digital Filter Design 46-51

46. Sketch the following specifications of a digital filter on paper. Which of the amplitude responses of the realizations in Figure 15 do fulfill the specifications?

Specifications: Digital lowpass filter, sampling frequency f_T 8000 Hz, passband edge frequency f_p 1000 Hz, transition band 500 Hz (transition band is the band between passband and stopband edge frequencies!), maximum passband attenuation 3 dB, minimum stopband attenuation 40 dB.

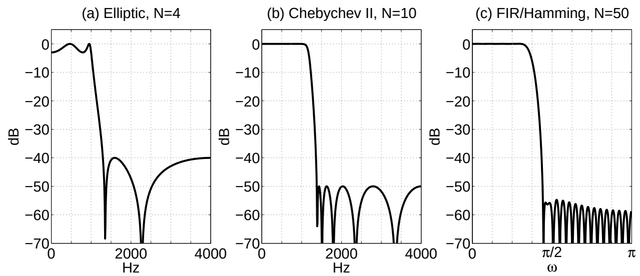


Figure 15: Three realizations in Problem 46: amplitude responses of (a) 4th order elliptic, (b) 10th order Chebychev II, (c) 50th order FIR using Hamming window.

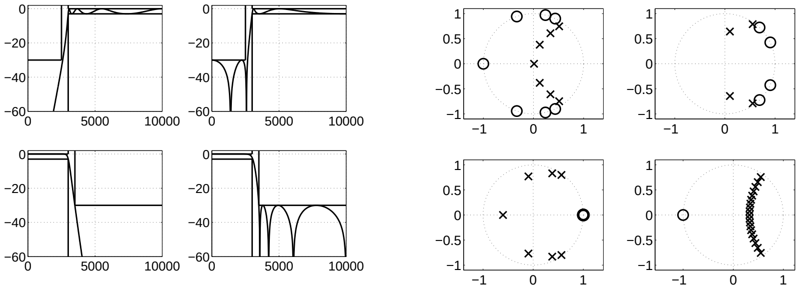
47. Connect first each amplitude response to the corresponding pole-zero plot in Figure 16. Then recognize the following digital IIR filter algoritms: Butterworth, Chebyshev I, Chebyshev II, Elliptic. The conversion from analog to digital form is done using bilinear transform. The sampling frequency in figures is 20 kHz.

48. Magnitude specifications are normally expressed in normalized form. The maximum of the amplitude response is scaled to one, and the frequency axis is scaled up to half of the sampling frequency, $0 \ldots \pi$. The first term of the denominator polynomial should also be 1.

Consider the following digital **lowpass** filter of type Chebyshev II:

$$H(z) = K \cdot \frac{0.71 - 0.36z^{-1} - 0.36z^{-2} + 0.71z^{-3}}{1 - 2.11z^{-1} + 1.58z^{-2} - 0.40z^{-3}}$$

Normalize the maximum of the amplitude response to the unity (0 dB).



(a) (b)

Figure 16: Problem 47. Digital filters from analog approximations through bilinear transform, (a) amplitude responses with specifications, $f_s = 20000$ Hz (b) pole-zero plots.

49. Consider the following prototype analog Butterworth-type lowpass filter

$$H_{protoLP}(s) = \frac{1}{s+1}$$

a) Form an analog first-order lowpass filter with cutoff frequency Ω_c by substituting $H(s) = H_{protoLP}(\frac{s}{\Omega_c})$. Draw the pole-zero plot in s-plane.

b) Implement a discrete first-order lowpass filter $H_{Imp}(z)$, whose cutoff frequency (-3 dB) is at $f_c = 100$ Hz and sampling rate is $f_s = 1000$ Hz, applying the impulse-invariant method to $H(s)$. Draw the pole-zero plot of the filter $H_{Imp}(z)$.

c) Implement a discrete first-order lowpass filter $H_{Bil}(z)$ with the same specifications applying the bilinear transform to $H(s)$. Prewarp the edge frequency. Draw the pole-zero plot of the filter $H_{Bil}(z)$.

50. Use windowed Fourier series method and design a FIR-type (causal) lowpass filter with cutoff frequency $3\pi/4$. Let the order of the filter be 4.

See Figure 17, in left the amplitude response of the ideal lowpass filter $H(e^{j\omega})$ with cut-off frequency at $3\pi/4$. In right, the corresponding inverse transform of the desired ideal filter $h_d[n]$, which is sinc-function according to the transform pair rect(.) $\leftrightarrow$ sinc(.):

$$h_d[n] = \{\ldots, -0.1592, 0.2251, \underline{0.75}, 0.2251, -0.1592, \ldots\}$$

a) Use the rectangular window of length 5, see Figure 18(a). The window function is $w_r[n] = 1, -M \le n \le M, M = 2$

b) Use the Hamming window of length 5, see Figure 18(b). The window function is

$$w_h[n] \;=\; 0.54 + 0.46\cos\left(\frac{2\pi n}{2M}\right), \quad -M \le n \le M, M = 2$$

which results to $w_h[n] = \{0.08, 0.54, \underline{1}, 0.54, 0.08\}$

c) Compare how the amplitude responses of the filters designed in (a) and (b) differ assuming that the window size is high enough (e.g. $M = 50$).

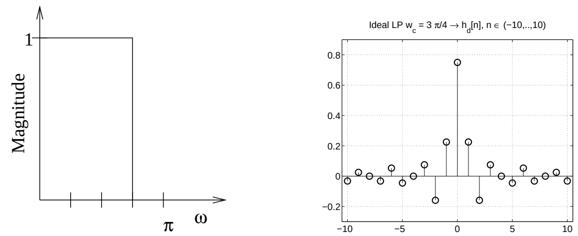


Figure 17: Problem 50: (a) The amplitude response of the ideal lowpass filter, and (b) the corresponding impulse response $h[n]$ values. The cut-off frequency is at $\omega_c = 3\pi/4$.

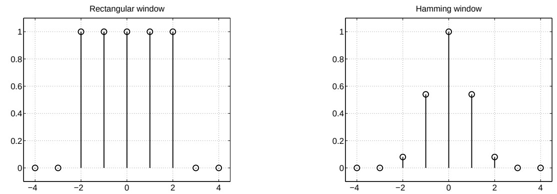


Figure 18: Problem 50: (a) rectangular window $w_r[n]$ of length 5, and (b) Hamming window $w_h[n]$ of length 5.

51. The following transfer functions $H_1(z)$ and $H_2(z)$ representing two different filters meet (almost) identical amplitude response specifications

$$H_1(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

where $b_0 = 0.1022$, $b_1 = -0.1549$, $b_2 = 0.1022$, $a_1 = -1.7616$, and $a_2 = 0.8314$, and

$$H_2(z) = \sum_{k=0}^{12} h[k] z^{-k}$$

where $h[0] = h[12] = -0.0068$, $h[1] = h[11] = 0.0730$, $h[2] = h[10] = 0.0676$, $h[3] = h[9] = 0.0864$, $h[4] = h[8] = 0.1040$, $h[5] = h[7] = 0.1158$, $h[6] = 0.1201$.

For each filter,

a) state if it is a FIR or IIR filter, and what is the order

b) draw a block diagram and write down the difference equation

c) determine and comment on the computational and storage requirements

d) determine first values of $h_1[n]$

Implementation Issues 52-56

52. Suppose that the calculation of FFT for a one second long sequence, sampled with 44100 Hz, takes 0.1 seconds. Estimate the time needed to compute (a) DFT of a one second long sequence, (b) FFT of a 3-minute sequence, (c) DFT of a 3-minute sequence. The complexities of DFT and FFT can be approximated with $\mathcal{O}(N^2)$ and $\mathcal{O}(N \log_2 N)$, respectively.

53. Express the desimal number -0.3125 as a binary number using sign bit and four bits for the fraction in the format of (a) sign-magnitude, (b) ones' complement, (c) two's complement. What would be the value after truncation, if only three bits are saved.

54. In the following Figure 19, some error probability density functions of the quantization error are depicted.

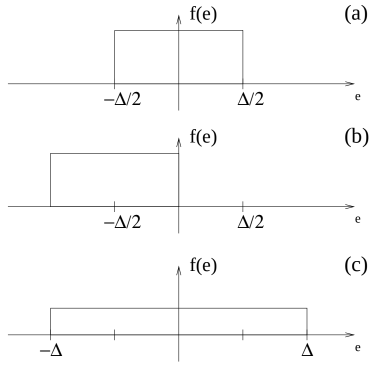


Figure 19: Problem 54: Error density functions.

(a) Rounding
(b) Two's complement truncation
(c) Magnitude (one's complement) truncation

is used to truncate the intermediate results. Calculate the expectation value of the quantization error m_e and the variance σ_e^2 in each case.
$\mathrm{E}[E] = \int_{-\infty}^{\infty} f(e)\, e \,\mathrm{d}e$, $\mathrm{Var}[E] = \mathrm{E}[(E - \mathrm{E}[E])^2] = \mathrm{E}[E^2] - (\mathrm{E}[E])^2$

55. In this problem we study the roundoff noise in direct form FIR filters. Consider an FIR filter of length N having the transfer function

$$H(z) = \sum_{k=0}^{N-1} h[k] z^{-k}.$$

Sketch the direct form realization of the transfer function.

a) Derive a formula for the roundoff noise variance when quantization is done before summations.
b) Repeat (a) for the case where quantization is done after summations, i.e. a double precision accumulator is used.

56. The quantization errors produced in digital systems may be compensated by error-shaping filters (*Mitra 2Ed Sec. 9.10 / 3Ed Sec. 12.10*). The error components are extracted from the system and processed e.g. using simple digital filters. This way the noise at the output of the system can be reduced.

Consider a lowpass DSP system with a second-order noise reduction system in Figure 20.

a) What is the transfer function of the system if infinite wordlength is used?

b) Derive an expression for the transform of the quantized output, $Y(z)$, in terms of the input transform, $X(z)$, and the quantization error, $E(z)$, and hence show that the error feedback network has no adverse effect on the input signal.
c) Deduce the expression for the error feedback function.
d) What values k_1 and k_2 should have in order to work as an error-shaping system?

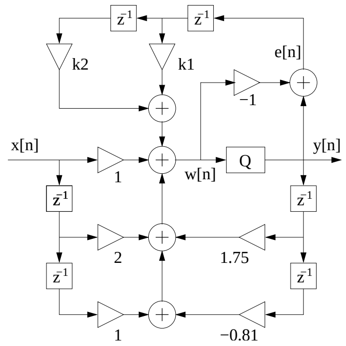


Figure 20: Second-order system with second-order noise reduction in Problem 56.

Multirate Systems 57-60

57. Consider a cosine sequence $x[n] = \cos(2\pi(f/f_s)n)$ where $f = 10$ Hz and $f_s = 100$ Hz as depicted in the top left in Figure 21. While it is a pure cosine, its spectrum is a peak at the frequence $f = 10$ Hz (top middle) or at $\omega = 2\pi f/f_s = 0.2\pi$ (top right).

a) Sketch the output sequence $x_u[n]$ with circles using up-sampler with up-sampling factor $L = 2$, and draw its spectra into second row. Original sequence values of $x[n]$ are marked with crosses. The spectrum in middle column is 0..200 Hz and in right $0..2\pi$, i.e., $0..f_s$.

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases} \qquad X_u(e^{j\omega}) = X(e^{j\omega L})$$

b) Sketch the output sequence $x_d[n]$ with circles using down-sampler with down-sampling factor $M = 2$, and draw its spectra into bottom row.

$$x_d[n] = x[nM] \qquad X_d(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})$$

58. Express the output $y[n]$ of the system shown in Figure 119 as a function of the input $x[n]$.

59. Show that the factor-of-L up-sampler $x_u[n]$ and the factor-of-M down-sampler $x_d[n]$ defined as in Problem 57 are linear systems.

60. Consider the multirate system shown in Figure 23 where $H_0(z)$, $H_1(z)$, and $H_2(z)$ are ideal lowpass, bandpass, and highpass filters, respectively, with frequency responses shown in Figure 24(a)-(c). Sketch the Fourier transforms of the outputs $y_0[n]$, $y_1[n]$, and $y_2[n]$ if the Fourier transform of the input is as shown in Figure 24(d).

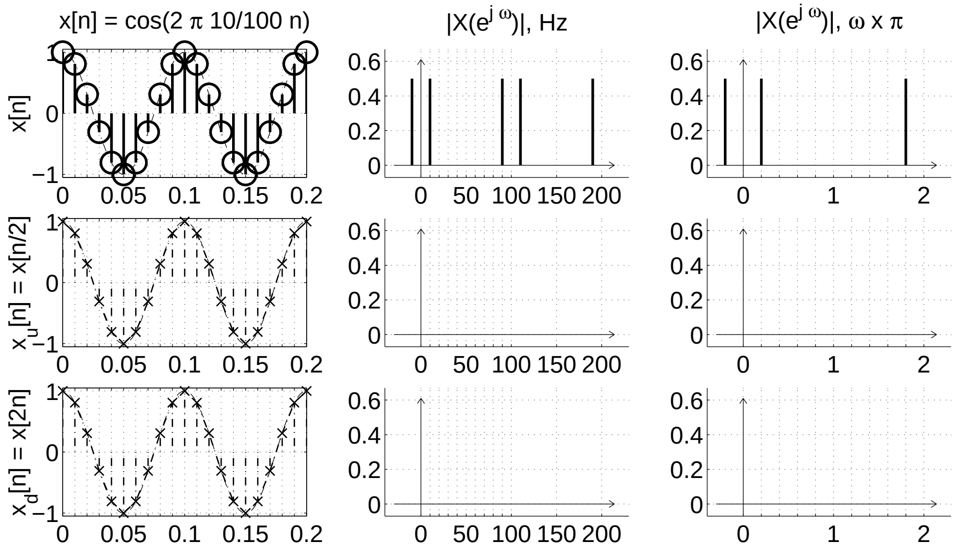


Figure 21: Empty figures for Problem 57. The up-sampling factor $L = 2$, and the down-sampling factor $M = 2$. **Left column**: sequence $x[n]$ with circles, fill in the sequences $x_u[n]$ and $x_d[n]$. X-axis: time ($0 \dots 0.2$ s). **Middle column**: Spectrum $X(e^{jf})$ (10 Hz component, 100 Hz sampling frequency), fill in the spectra $X_u(e^{jf})$ and $X_d(e^{jf})$. X-axis: frequency ($0 \dots 200$ Hz). **Right column**: Spectrum $X(e^{j\omega})$ ($2\pi \cdot (10/100) = 0.2\pi$), fill in the spectra $X_u(e^{j\omega})$ and $X_d(e^{j\omega})$. X-axis: angular frequency ($0 \dots 2\pi$).

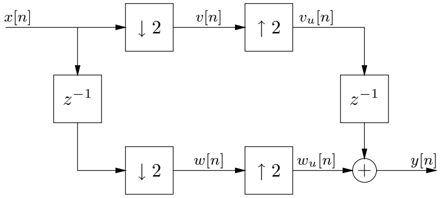


Figure 22: Multirate system of Problem 58.

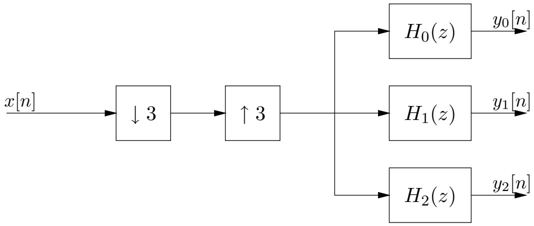


Figure 23: Multirate system of Problem 60.

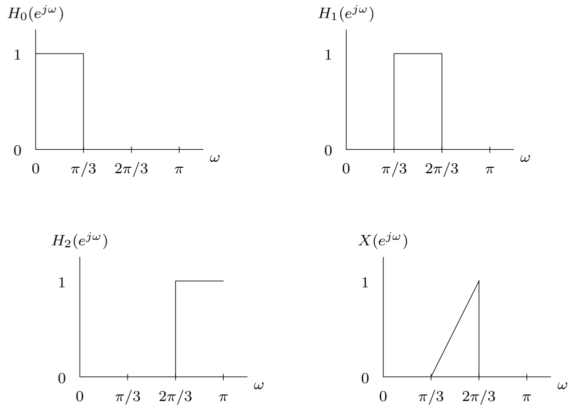


Figure 24: (a)-(c) Ideal filters $H_0(z)$, $H_1(z)$, $H_2(z)$, (d) Fourier transform of the input of Problem 60.

T-61.3010 Digital Signal Processing and Filtering

Solutions for example problems for spring 2006.
Corrections and comments to `t613010@cis.hut.fi`, thank you!

Solutions

1. **Problem:**

   a) Express $z = 2e^{-j\pi}$ in rectangular coordinates.

   b) Express $z = -1 + 2j$ in polar coordinates.

   c) Which (two) angles satisfy $\sin(\omega) = 0.5$?

   d) What are $z + z^*$, $|z + z^*|$? and $\angle(z + z^*)$? What are zz^*, $|zz^*|$? and $\angle zz^*$?

   Solution:

   a) "Brute force" using Euler's formula and $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$,

   $$z = 2e^{-j\pi} = 2(\cos(-\pi) + j\sin(-\pi)) = 2(\cos(\pi) - j\sin(\pi)) = -2$$

   or using directly the unit circle and seeing that when the angle is $-\pi$ in radians (-180 degrees) then $e^{-j\pi} = -1$.

   b) The radius $r = \sqrt{(-1)^2 + 2^2} = \sqrt{5} \approx 2.2$ and the angle in radians $\theta = \pi - \arctan(2/1) \approx 2.03 \approx 0.65\pi$. So, $z = -1 + 2j = \sqrt{5}\, e^{j(\pi - \arctan(2))} \approx 2.2\, e^{2.03j}$. Note! Always check the right quarter in the figure.

   c) From Figure 25, $\omega_1 = \arcsin(0.5) = \pi/6$ and $\omega_2 = \pi - \arcsin(0.5) = 5\pi/6$

   d) Summing can be graphically considered as concatenation of vectors. $z + z^* = r(e^{j\omega} + e^{-j\omega}) = 2r\cos(\omega) \in \mathbb{R}$. From previous, $|z + z^*| = |2r\cos(\omega)|$ and $\angle(z + z^*) = 0$. Using Carthesians, $z + z^* = 2x$.

   Product of complex number and its complex conjugate: $zz^* = (re^{j\omega})(re^{-j\omega}) = r^2 e^{j(\omega-\omega)} = r^2$, and $|zz^*| = r^2$ and $\angle zz^* = 0$.

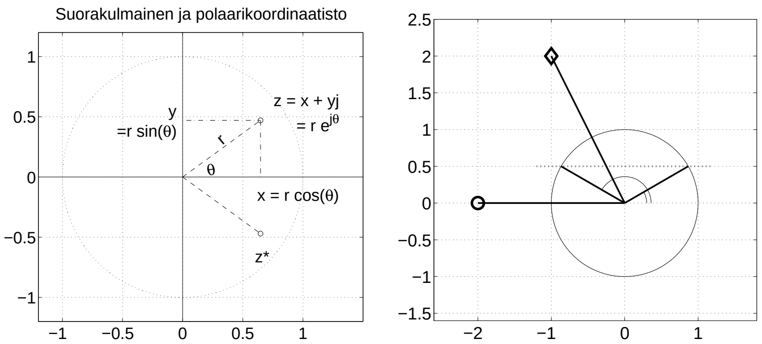


Figure 25: Problem 1, unit circle in complex plane (left), and points for (a), (b), and (c) (right).

2. **Problem:** Euler's formula is $e^{j\theta} = \cos(\theta) + j \cdot \sin(\theta)$. Express with cosines and sines: (a) $e^{j\theta} + e^{j(-\theta)}$, (b) $e^{j\theta} - e^{j(-\theta)}$, (c) $e^{j\pi/8} \cdot e^{j\theta} - e^{j(-\pi/8)} \cdot e^{j(-\theta)}$.

   Solution: Euler's formula $e^{j\theta} = \cos(\theta) + j \cdot \sin(\theta)$ can be thought as a phasor going round on the unit circle. It is unit circle because $|e^{j\theta}| = \sqrt{\cos^2 + \sin^2} = 1$ always. Real part of $e^{j\theta}$ is cosine, and imaginary part is sine.

   a) Sum of exponentials at positive frequency θ and negative frequency $-\theta$ gives a real cosine at frequency θ.

   $$\begin{aligned} e^{j\theta} &= \cos(\theta) + j \cdot \sin(\theta) \\ e^{j(-\theta)} &= \cos(-\theta) + j \cdot \sin(-\theta) \\ &= \cos(\theta) - j \cdot \sin(\theta) \end{aligned}$$

   Adding the first and list row we get

   $$e^{j\theta} + e^{j(-\theta)} = 2\cos(\theta) \in \mathbb{R}$$

   b) In the same way as in (a)

   $$\begin{aligned} e^{j\theta} &= \cos(\theta) + j \cdot \sin(\theta) \\ e^{j(-\theta)} &= \cos(-\theta) + j \cdot \sin(-\theta) \\ &= \cos(\theta) - j \cdot \sin(\theta) \end{aligned}$$

   Substracting the last from the first gives

   $$e^{j\theta} - e^{j(-\theta)} = 2j\sin(\theta) \in \mathbb{C}$$

   which is pure complex. In other words, cosine and sine are:

   $$\begin{aligned} \cos(\theta) &= 0.5 \cdot e^{j\theta} + 0.5 \cdot e^{j(-\theta)} \\ \sin(\theta) &= -0.5j \cdot e^{j\theta} + 0.5j \cdot e^{j(-\theta)} \end{aligned}$$

   where $1/(2j) = -j/2$ as shown in Problem 3(e).

   c) This can be thought as phase shift. First, use the rule $e^x \cdot e^y = e^{x+y}$,

   $$\begin{aligned} e^{j\theta} \cdot e^{j\pi/8} &= e^{j(\theta+\pi/8)} \\ e^{-j\theta} \cdot e^{-j\pi/8} &= e^{-j(\theta+\pi/8)} \end{aligned}$$

   Now, we see using (b)

   $$e^{j\pi/8} \cdot e^{j\theta} - e^{j(-\pi/8)} \cdot e^{j(-\theta)} = 2j\sin(\theta + \pi/8)$$

   Notice that each real cosine with positive angle and each real sine with positive angle can be replaced by two complex exponentials with positive and negative angles. When considering Fourier analysis, the real cosine signal with frequency f_c can be represented in the spectrum with a peak at f_c (in one-side spectrum) or with peaks at f_c and $-f_c$ (in two-side spectrum). Vice versa, if the two-side spectrum is not symmetric, then the signal is not real but complex. More about this later in Fourier analysis.

3. **Problem:** Consider the following three complex numbers $z_1 = 3+2j$, $z_2 = -2+4j$, and $z_3 = -1-5j$. (a) Draw the vectors z_1, z_2, and z_3 separately in complex plane. (b) Draw and compute the sum $z_1+z_2+z_3$. (c) Draw and compute the weighted sum $z_1-2z_2+3z_3$. (d) Draw and compute the product $z_1 \cdot z_2 \cdot z_3$. (e) Compute and reduce the division z_1/z_2.

Solution:

a) Each number can be thought as a vector starting from origo and the other end at point z. See Figure 26.

b) Real parts and imaginary parts can be summed separately $z = (3-2-1)+(2+4-5)j = -j$. This can be expressed in polar coordinates $z = e^{j(-\pi/2)}$, i.e. on unit circle (radius 1) and the angle one fourth a circle clockwise.

c) If you are computing without computer, be attentive and check twice that all coefficients are correctly reduced. $z = (3+2j)-2(-2+4j)+3(-1-5j) = 3+2j+4-8j-3-15j = 4-21j$. Again, in polar coordinates $r = \sqrt{(4)^2+(-21)^2} \approx 21.38$. The angle $\theta = \arctan((-21)/(4)) \approx -1.38 \approx -0.44\pi$.
If $z = -4-21j$, then $\theta = \arctan((-21)/(-4)) \approx -\pi + 1.38 \approx -1.76 \approx -0.56\pi$. Notice that now z is in the third quarter, so the angle 1.38 that calculator gives is NOT the correct answer.

d) When using rectangular coordinates, multiply terms normally, $j^2 = -1$. The product in polar coordinates means multipling the lengths of vectors and summing the angles.

$$\begin{aligned} z &= ((3+2j)\cdot(-2+4j))\cdot(-1-5j) \\ &= (-14+8j)\cdot(-1-5j) \\ &= 54+62j \\ &= \sqrt{9+4}\cdot\sqrt{4+16}\cdot\sqrt{1+25}\cdot e^{j(\arctan(2/3)+\arctan(4/(-2))+\arctan((-5)/(-1)))} \\ &\approx 82.2\cdot e^{j(0.27\pi)} \end{aligned}$$

e) The denominator is now complex. If both sides are multiplied by the complex conjugate then the denominator becomes real. Just as in Problem 1 $z\cdot z^* = |z|^2 = r^2 \in \mathbb{R}$. Notice also that $1/j$ is $-j$ $((1/j)\cdot(j/j) = j/j^2)$.

$$\begin{aligned} z &= (3+2j)/(-2+4j) \qquad |\cdot(-2-4j)/(-2-4j) \\ &= (2-12j)/20 \end{aligned}$$

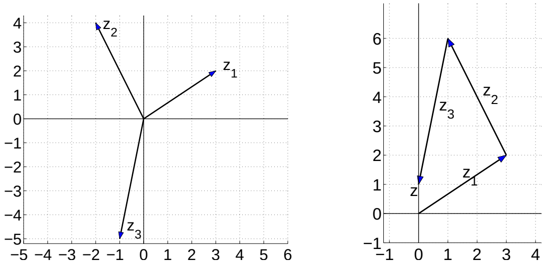


Figure 26: The vectors in Problem 3(a) and (b).

4. **Problem:**

Examine a complex-valued function

$$H(\omega) = 2 - e^{-j\omega}$$

where $\omega \in [0\ldots\pi] \in \mathbb{R}$.

a) Compute values of Table 4 with a calculator. Euler: $e^{j\omega} = \cos(\omega) + j\,\sin(\omega)$.

b) Draw the values at $\omega = \{0,\ \pi/4,\ \ldots,\ \pi\}$. Interpolate.

c) Sketch $|H(\omega)|$ as a function of ω. Interpolate.

d) Skecth $\angle H(\omega)$ as a function of ω. Interpolate.

Solution: In this course complex-valued functions are widely used, e.g. as frequency responses of the systems or in Fourier transforms. The argument of the function is real-valued $\omega \in \mathbb{R}$, but the value of the function is (normally) complex $H(\omega) \in \mathbb{C}$ due to complex factor $e^{j\omega}$. In case of the transfer function $H(z)$ both z and $H(z)$ are complex-valued.

a) Sometimes it is possible to simplify $H(\omega)$. However, normally it is useful to write down a suitable format for the use of the calculator. In this case, Carthesian coordinate system with x and y is used:

$$\begin{aligned} H(\omega) &= 2 - e^{-j\omega} = 2 - (\cos(-\omega) + j\,\sin(-\omega)) \\ &= \underbrace{2-\cos(\omega)}_{x} + j\,\underbrace{\sin(\omega)}_{y} \end{aligned}$$

The variables r and θ of the polar coordinate system are received from the right-angled triangle: $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$.
On the other hand, in this case it is easily seen that there is only a circle ($e^{-j\omega}$) whose origin is at $z = 2$.

ω	$x = \mathrm{Real}(H(\omega))$	$y = \mathrm{Imag}(H(\omega))$	$r = \lvert H(\omega)\rvert$	$\theta = \angle H(\omega)$
0	1.0000	0	1.0000	0
$\pi/4$	1.2929	0.7071	1.4736	0.1593π
$\pi/2$	2.0000	1.0000	2.2361	0.1476π
$3\pi/4$	*2.7071*	*0.7071*	*2.7979*	*0.0813π*
π	3.0000	0	3.0000	0

Table 4: Problem 8: values of a complex-valued function in rectangular (x, y) and polar (r, θ) coordinates. The row $3\pi/4$ is highlighted for Figure 27.

b) Take the columns x and y of Table 4 and sketch the curve like in Figure 27(left). There is a line drawn in the plot, from the origo to a point related to $\omega = 3\pi/4$, i.e. (x, y). The length of the line is r and the angle between the line and x-axis is θ, so it can be written in polar coordinates $r\,e^{j\theta}$.

c) Take the column r of Table 4 and sketch the curve like in Figure 27(middle). The plot shows the distance r from the origo to a point at given value of ω.

d) Take the column θ of Table 4 and sketch the curve like in Figure 27(right). The plot shows the angle θ between the origo and a point at given value of ω.

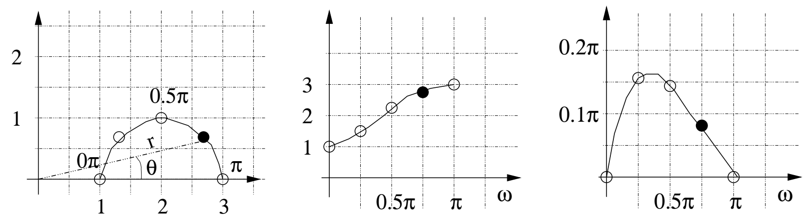


Figure 27: Problem 8: Plots of a complex-valued function. Left, $H(\omega)$ in complex plane; middle, absolute values $|H(\omega)|$; and right, angle $\angle H(\omega)$. The case when $\omega = 3\pi/4$ is highlighted.

5. **Problem:**

   a) Estimate A, f, θ for the cosine $x_1(t)$ in Figure 28(a).

   b) Sketch a cosine $x_2(t)$, with $A = 2$, angular frequency 47 rad/s and angle $-\pi/2$.

   c) Express $x_2(t)$ in (b) using exponential functions.

Solution: There are a lot of variation in symbols in different signal processing books and texts. There are probably also variation in these exercises. However, we try to use the following symbols listed in Table 5.

symbol	units	meaning
f	Hz	frequency
Ω	rad/s	angular frequency, $\Omega = 2\pi f$
ω	rad	normalized angular frequency, $\omega = 2\pi(\Omega/\Omega_s)$
f_{MATLAB}	1	normalized Matlab frequency, $f_{MATLAB} = 2f/f_s$

Table 5: Problem 5, symbols of frequencies. f_s refers to sampling frequency, and $\Omega_s = 2\pi f_s$.

A cosine signal can be represented using its angular frequency Ω or frequency f, amplitude A and phase θ:

$$x(t) = A\,\cos(\Omega t + \theta) = A\,\cos(2\pi f t + \theta)$$

For a discrete sequence of numbers

$$x[n] = x(t)|_{t=nT_s} = x(t)|_{t=n/f_s} = A\,\cos(2\pi(f/f_s)n + \theta) = A\,\cos(\omega n + \theta)$$

where T_s is sampling interval (period), f_s sampling frequency, and ω (normalized) angular frequency.

a) Cosine oscillates between -0.8 and 0.8, so $A = 0.8$. There is no phase shift, $\theta = 0$. There is one oscillation in 0.2 seconds, so there are 5 periods in one second, $f = 5$ Hz, or $\Omega = 2\pi f = 10\pi$ rad/s.
   Hence, $x_1(t) = 0.8\cos(10\pi t)$.

b) $x_2(t)$ can be written directly $x_2(t) = 2\,\cos(47t - \pi/2)$.
   If $\Omega = 47$ rad/s, then $f \approx 7.5$ Hz. In 0.1 seconds there are 0.75 periods. At $t = 0$, $x_2(0) = 2\cos(-\pi/2) = 0$, and increasing. Note that $\cos(\Omega t - \pi/2) \equiv \sin(\Omega t)$. The curve is plotted in Figure 28(b).

c) Using Euler's formula, and properties of cosine (even function $f(-x) = f(x)$) and sine (odd function $f(-x) = -f(x)$),

$$\begin{array}{rrcl} & e^{j\omega} & = & \cos(\omega) + j\,\sin(\omega) \\ + & e^{-j\omega} & = & \cos(\omega) - j\,\sin(\omega) \\ \hline & e^{j\omega} + e^{-j\omega} & = & 2\cos(\omega) \end{array}$$

$$\begin{array}{rrcl} & e^{j\omega} & = & \cos(\omega) + j\,\sin(\omega) \\ + & -\,e^{-j\omega} & = & -\cos(\omega) + j\,\sin(\omega) \\ \hline & e^{j\omega} - e^{-j\omega} & = & 2j\,\sin(\omega) \end{array}$$

Now, it can be seen that

$$\begin{aligned} x_2(t) &= 2\cos(47t - \pi/2) \\ &= e^{j(47t-\pi/2)} + e^{-j(47t-\pi/2)} \end{aligned}$$

which can be even "simplified" to $x_2(t) = j[e^{-j47t} - e^{j47t}]$.

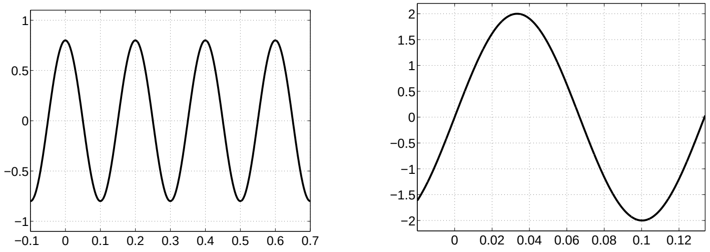


Figure 28: Cosine $x_1(t)$ (left) and $x_2(t)$ (right) in Problem 5.

6. **Problem:**

   a) Compute with a calculator: $\log_8 7$.

   b) The power of signal is attenuated from 10 to 0.01. How much is the attenuation in decibels?

   c) Sketch the curve $p(x) = \sum_{k=-N}^{+N} kx$ for various N.

   d) Consider $h(n) = \sin(0.75\pi n)/(\pi n)$. What is $h(0)$?

   e) Modulo-N operation for number x is written here as $< x >_N$. What is $< -4 >_3$?

   f) What is the binary number $(1001011)_2$ as a decimal number?

Solution:

a) $\log_8 7 = \log_e 7/\log_e 8 \approx 1.9459/2.0794 \approx 0.936$.
Sometimes it is useful to convert, e.g., 2^{2006} to decimal base: $2^{2006} = 10^x$, taking $\log_{10}$ on both sides: $x = 2006\log_{10} 2 \approx 603.8662$. Now $10^{0.8662} \approx 7.3485$, which finally gives $2^{2006} \approx 7.3 \cdot 10^{603}$.

b) Decibel scales are widely used to compare two quantities. The decibel difference between two power levels, ΔL, is defined in terms of their power ratio W_2/W_1 (p. 99, Rossing et al., The Science of Sound, 3rd Edition, Addison Wesley)

$$\Delta L = L_2 - L_1 = 10\log_{10} W_2/W_1$$

Now the power (square) of signal is attenuated from 10 to 0.01, so the signal is attenuated by 30 dB:

$$10\log_{10}(0.01/10) = 10\log_{10} 10^{-3} = -30$$

In case of computing amplitude response $|H(e^{j\omega})|$, e.g. in Matlab directly from the equation or with the command `freqz`, the values are squared for decibels

$$10\log_{10}|(H/H_0)|^2 = 20\log_{10}|(H/H_0)|$$

c) If Σ confuses, open the expression! There is hardly anything to draw!

$$\begin{aligned} p(x) &= \sum_{k=-N}^{+N} kx = (-N)x + \ldots + (-2)x + (-1)x + 0x + x + 2x + \ldots + Nx \\ &\equiv 0, \qquad \forall N, x \end{aligned}$$

d) Sinc-function is very useful in the signal processing, and it is defined $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. Also it is known that $\sin(x)/x \to 1$, when $x \to 0$, and with sinc-function $\mathrm{sinc}(0) = 1$. Fourier-transform of a rectangular (box) signal produces spectrum with shape of sinc-function, and vice versa, a signal like sinc-function has a spectrum of rectangular (box) shape.
Note that the result of the problem is not 1 nor 0,

$$\begin{aligned} h(n) &= \sin(0.75\pi n)/(\pi n) = 0.75\sin(0.75\pi n)/(0.75\pi n) \\ h(0) &\to 0.75 \end{aligned}$$

e) Normally we want to deal with numbers $0 \ldots N-1$. Modulo-N with $N = 3$ means simply that $< x >_N = x + kN, \forall k$,

$$\ldots = < -4 >_3 = < -1 >_3 = < 2 >_3 = \ldots$$

A circular buffer is implemented in the instruction sets of many DSPs. Assume that there is a buffer of size 1024 bytes, with addresses $0x0000$ to $0x03FF$ in hexadecimals. New 8-bit (byte) samples are read into a buffer where an address counter (pointer) is increased by one each time. When the counter has the value $0x03FF$, the next value is $< 0x0400 >_{0x0400} = 0x0000$. In other words, the oldest sample is replaced by the newest. See Figure 29 for figures of linear and circular buffers.

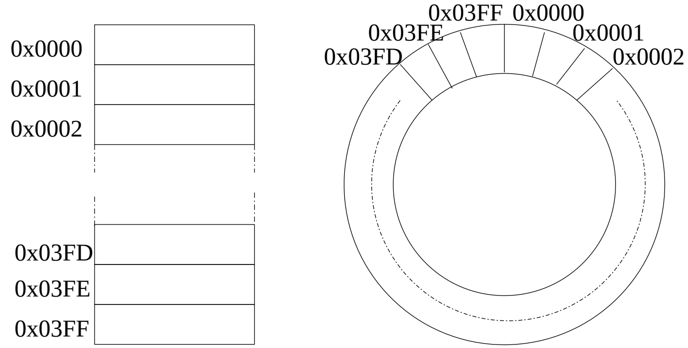


Figure 29: Problem 6: linear and circular buffer.

f) The result depends on which number representation is chosen. In case of multi-byte data types numbers can be saved in big-endian or little-endian manner. DSPs are divided to fixed-point and floating-point processors (IEEE 754, sign bit, exponent and mantissa fields). Least significant bit (LSB) is normally the last bit, most significant bit (MSB) leftmost. Negative numbers and fractions has to be considered, too. (*Mitra 2Ed Sec. 8.4 / 3Ed Sec. 11.8*) deals with all aspects of the number representation.
When both negative and positive b-bit fraction values are needed, 1001011 is considered to have a sign bit first, and then fraction bits, like $s_\Delta a_{-1}a_{-2}\ldots a_{-b}$. Table 6 contains some possible results with values $b = 6$ and $s = 1$, see also (*Mitra 2Ed Table 8.1, p. 557 / 3Ed Table 11.1, p. 638*).

non-negative fixed-point	1001011	$1\cdot 64 + 1\cdot 8 + 1\cdot 2 + 1\cdot 1 = 75$
sign-magnitude	$1_\Delta 001011$	$(-2s+1)\sum_{i=1}^{b} a_{-i}2^{-i} = -11/64 \approx -0.1719$
ones' complement	$1_\Delta 001011$	$-s\cdot(1-2^{-b}) + \sum_{i=1}^{b} a_{-i}2^{-i} = -52/64 \approx -0.8125$
two's complement	$1_\Delta 001011$	$-s + \sum_{i=1}^{b} a_{-i}2^{-i} = -53/64 \approx -0.8281$
offset binary	$1_\Delta 001011$	$+11/64 \approx +0.1719$

Table 6: Problem 6: Examples on binary number representations with values $b = 6$ and $s = 1$.

7. **Problem:**

a) Compute roots of $H(z) = z^2 + 2z + 2$.

b) Compute roots of $H(z) = 1 + 16z^{-4}$.

c) Compute long division $(4z^4 - 8z^3 + 3z^2 - 4z + 6)/(2z - 3)$.

Solution: In this course roots of transfer function $H(z)$ provide information on the behaviour of the filter. The order of the rational polynomial $H(z) = B(z)/A(z)$ is the maximum of the orders of $B(z)$ and $A(z)$.

a) The order of $H(z)$ is 2. Using the equation for solving the second-order polynomials $z = (-b \pm \sqrt{b^2 - 4ac})/(2a)$, the roots are $z_1 = -1 + j$ and $z_2 = -1 - j$. This can be assured by multiplication $(z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2 = z^2 + 2z + 2$.

b) The order of $H(z)$ is 4. Now, when setting $H(z) = 1 + 16z^{-4} = 0$, the equation can be multiplied by z^4 on both sides. Hence, $z^4 + 16 = 0$ and $z = \sqrt[4]{-16}$. Because $-16 = 2^4 \cdot e^{j(\pi + 2\pi k)}$, we get four roots using $\sqrt[N]{z} = |\sqrt[N]{r}| \cdot e^{j(2\pi k/N + \theta/N)}$.

Roots: $z_k = 2\, e^{j(2\pi k/4 + \pi/4)}$, with $k = 0 \ldots 3$. Again, $z_1^4 = (2e^{j\pi/4})^4 = 2^4 e^{j4\pi/4} = 16e^{j\pi} = -16$, and similarly other z_k result to -16. In Figure 30 all four roots are plotted with circles.

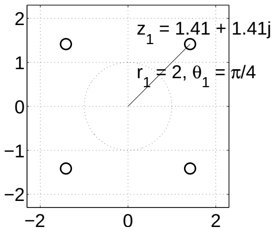


Figure 30: Problem 7(b): four roots of $H(z) = 1 + 16z^{-4}$.

c) Division operation can be applied to polynomials just as for normal numbers. Polynomial product and division have a very close connection to the convolution operation. For example, in Matlab there is the same function `conv` for the both operations.

$$
\begin{array}{r|l}
 & 2z^3 - z^2 \qquad - 2 \\
\hline
2z - 3) & 4z^4 - 8z^3 + 3z^2 - 4z + 6 \\
 & -4z^4 + 6z^3 \\
\hline
 & -2z^3 + 3z^2 \\
 & 2z^3 - 3z^2 \\
\hline
 & -4z + 6 \\
 & 4z - 6 \\
\hline
 & 0
\end{array}
$$

8. **Problem:** Examine a complex-valued function ($z \in \mathbb{C}$)

$$H(z) = \frac{1 + 0.5z^{-1} + 0.06z^{-2}}{1 - 1.4z^{-1} + 0.48z^{-2}}$$

(a) Multiply both sides by z^2. (b) Solve $z^2 + 0.5z + 0.06 = 0$. (c) Solve $z^2 - 1.4z + 0.48 = 0$. (d) $H(z)$ can be written: $H(z) = K \cdot ((z - z_1) \cdot (z - z_2)) \,/\, ((z - p_1) \cdot (z - p_2))$. What are the five values? (e) What are the coefficients of $H(z)$. What are the roots of $H(z)$? What is the order of the numerator polynomial of $H(z)$? What is the order of the denominator polynomial of $H(z)$?

Solution: In this course complex-valued functions are widely used. In case of the transfer function $H(z)$ both z and $H(z)$ are complex-valued. A typical form of a transfer function of a FIR filter is

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}$$

and that of an IIR filter is

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

a) Multiplication $H(z) \cdot (z^2/z^2)$ does not change the values of $H(z)$, but it is more convenient to work with positive exponentials:

$$H(z) = \frac{z^2 + 0.5z + 0.06}{z^2 - 1.4z + 0.48}$$

b) Using the formula for second order polynomials $az^2 + bz + c = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we get easily the roots $z_1 = -0.3$, $z_2 = -0.2$. In Matlab you can write `P = [1 0.5 0.06]; roots(P)`.

c) Similarly, the roots $p_1 = 0.8$, $p_2 = 0.6$.

d) Using the notation from (b) and (c),

$$\begin{aligned} H(z) &= K \cdot \frac{(z + 0.3) \cdot (z + 0.2)}{(z - 0.8) \cdot (z - 0.6)} \\ &= K \cdot \frac{z^2 + 0.5z + 0.06}{z^2 - 1.4z + 0.48} \end{aligned}$$

we can scale $H(z)$ correctly by choosing $K = 1$.

e) In this case the coefficients were $\{1,\ 0.5,\ 0.06\}$ in numerator polynomial (upper part), and $\{1,\ -1.4,\ 0.48\}$ in denominator polynomial (bottom part).

Roots were computed in (b) and (c). In DSP we call the roots of numerator polynomial as "zeros". The roots of denominator polynomial (bottom part) are "poles".

As seen in (d) the same function $H(z)$ can be expressed either using coefficients or roots (and scaling factor). In the filter analysis the positions of roots give some information on the nature of the filter. More about this in Problem 32.

9. **Problem:**

a) Decompose $f(x) = 1/(x^2 + 1)$
b) Decompose $H(z) = (0.4 - 0.2z^{-1})/(1 - 0.1z^{-1} - 0.06z^{-2})$

Solution: In this course partial fractions are used when finding an explicit form of the impulse response $h[n]$ from the transfer function $H(z)$. In the list of Fourier-transform pairs there are only inverse transforms for the first order expressions. So, if the transfer function is of second-order or higher, it has to be converted to a sum of first-order expressions by partial fraction decomposition (expansion).

Decomposition requires taking roots of a polynomial, so it is possible to derive by hands only in some cases, e.g., $1/(x^2 + 3x + 2) = 1/(x + 1) - 1/(x + 2)$. For more complicated cases, see (*Mitra 2Ed Sec. 3.9 / 3Ed Sec. 6.4*), or any other math reference. When using Matlab, the command is `residuez`.

Rules of thumb, (1) compute roots of the denominator polynomial, (2) write down the sum of first-order rational polynomials, (3) compute the unknown constants (equation pairs). Note that the decomposition in not unique, but there are several different expressions which lead to the same result.

a) Find the roots of the denominator: $x^2 + 1 = 0 \Rightarrow x_1 = -j,\ x_2 = j$. Roots can be complex, too! Hence,

$$\begin{aligned} f(x) &= \frac{A}{x - x_1} + \frac{B}{x - x_2} = \frac{A}{x + j} + \frac{B}{x - j} \\ &= \frac{A(x - j) + B(x + j)}{x^2 + jx - jx + 1} = \frac{x(A + B) + j(-A + B)}{x^2 + 1} \end{aligned}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -A + B = -j \end{cases} \quad \Rightarrow \begin{cases} A = 0.5j \\ B = -0.5j \end{cases}$$

Finally,

$$f(x) = \frac{0.5j}{x + j} - \frac{0.5j}{x - j}$$

b) In this course z^{-1} corresponds a unit delay in time-domain. The numerator polynomial can divided and z^{-1} terms can be taken to front, and the partial fraction is done only once for $P(z)$, whose numerator polynomial is plain 1,

$$\begin{aligned} H(z) &= \frac{0.4 - 0.2z^{-1}}{1 - 0.1z^{-1} - 0.06z^{-2}} \\ &= 0.4 \cdot \underbrace{\frac{1}{1 - 0.1z^{-1} - 0.06z^{-2}}}_{P(z)} - 0.2z^{-1} \cdot \underbrace{\frac{1}{1 - 0.1z^{-1} - 0.06z^{-2}}}_{P(z)} \end{aligned}$$

The denominator of $P(z)$ is set to zero and multiplied by z^2: $z^2 - 0.1z - 0.06 = 0$. The roots are $z_1 = 0.3$ and $z_2 = -0.2$.

$$P(z) = \frac{A}{1 - 0.3z^{-1}} + \frac{B}{1 + 0.2z^{-1}} = \frac{A + 0.2Az^{-1} + B - 0.3Bz^{-1}}{1 - 0.1z^{-1} - 0.06z^{-2}}$$

Now we get a pair of equations $\begin{cases} A + B = 1 \\ 0.2A - 0.3B = 0 \end{cases} \Rightarrow \begin{cases} A = 0.6 \\ B = 0.4 \end{cases}$ and finally,

$$H(z) = 0.4 \cdot \left(\frac{0.6}{1 - 0.3z^{-1}} + \frac{0.4}{1 + 0.2z^{-1}}\right) - 0.2z^{-1} \cdot \left(\frac{0.6}{1 - 0.3z^{-1}} + \frac{0.4}{1 + 0.2z^{-1}}\right)$$

10. **Problem:**

a) What is sum of series $S = \sum_{k=0}^{\infty} (0.5)^k$.
b) $S = \sum_{k=10}^{\infty} (-0.6)^{k-2}$.
c) $S = \sum_{k=2}^{\infty} (0.8^{k-2} \cdot e^{-j\omega k})$.

Solution: Sum of geometric series is applied in Fourier- and z-transforms. When the ratio q in geometric series is $|q| < 1$, the sum of series converges to $\sum_{k=0}^{\infty} q^k = 1/(1 - q)$, and correspondingly $\sum_{k=0}^{N} q^k = (1 - q^{N+1})/(1 - q)$.

a) Directly from the formula with $q = 0.5$, $S = 1/(1 - 0.5) = 2$.

b) Open Σ expression if it seems to be difficult.

$$\begin{aligned} S &= \sum_{k=10}^{\infty} (-0.6)^{k-2} = (-0.6)^8 + (-0.6)^9 + (-0.6)^{10} + \ldots \\ &= \sum_{k=8}^{\infty} (-0.6)^k \\ &= \sum_{k=0}^{\infty} (-0.6)^k - \sum_{k=0}^{7} (-0.6)^k \\ &= 1/(1 + 0.6) - (1 - (-0.6)^8)/(1 + 0.6) = (-0.6)^8/1.6 \approx 0.0105 \end{aligned}$$

c) Discrete-time Fourier-transform is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$\begin{aligned} S &= \sum_{k=2}^{\infty} (0.8^{k-2} \cdot e^{-j\omega k}) \qquad |k = m + 2 \\ &= \sum_{m=0}^{\infty} (0.8^m \cdot e^{-j\omega m} \cdot e^{-j2\omega}) \\ &= e^{-j2\omega} \cdot \sum_{m=0}^{\infty} (0.8e^{-j\omega})^m \\ &= e^{-j2\omega} \cdot \frac{1}{1 - 0.8e^{-j\omega}} \end{aligned}$$

The term $e^{-j2\omega}$ can be seen as a time shift (delay) of two units.

11. **Problem:**

a) List all integral transforms that are used in previous signal processing courses.

b) Compute the integral $X(\Omega) = \int_0^4 e^{-j\Omega t}\mathrm{d}t$.

Solution: A general integral transform is defined by

$$F(\omega) = \int_a^b f(t)K(\omega, t)\mathrm{d}t$$

where $K(\omega, t)$ is an integral kernel of the transform, see e.g. (*Mitra 2Ed Sec. - / 3Ed Sec. 5.1*).

a) In our case the time-domain signal is transformed to the frequency-domain in order to improve the analyse. For example, the structure of a periodic signal can be seen easily in the spectrum.

Periodic signals can be represented as Fourier-series. Laplace- and z-transforms are more general than Fourier-transforms. There are versions for both analog and digital signals as well as for one-dimensional and two-dimensional signals. Certain transforms are used in particular applications, say, discrete cosine transform is used in JPEG and wavelet transform in JPEG2000.

b) Now $x(t)$ can be considered as a rectangular signal, and its Fourier transform is a sinc-function.

$$\begin{aligned} X(\Omega) &= \int_0^4 e^{-j\Omega t}\mathrm{d}t = \Big|_0^4 (1/(-j\Omega))e^{-j\Omega t} = (1/(-j\Omega))(e^{-j4\Omega} - 1) \\ &= (1/(-j\Omega))(-e^{-j2\Omega})(e^{j2\Omega} - e^{-j2\Omega}) = (1/(-j\Omega))(-e^{-j2\Omega})(2j\sin(2\Omega)) \\ &= 4e^{-j2\Omega}(\sin(2\Omega)/(2\Omega)) = 4e^{-j2\Omega}\,\mathrm{sinc}(2\Omega/\pi) \end{aligned}$$

12. **Problem:** The unit impulse function $\delta[n]$ and the unit step function $\mu[n]$ (or $u[n]$) are defined

$$\delta[n] = \begin{cases} 1, & \text{when } n = 0 \\ 0, & \text{when } n \neq 0 \end{cases} \qquad \mu[n] = \begin{cases} 1, & \text{when } n \geq 0 \\ 0, & \text{when } n < 0 \end{cases}$$

Sketch the following sequences around the origo (a) $x_1[n] = \sin(0.1\pi n)$, (b) $x_2[n] = \sin(2\pi n)$, (c) $x_3[n] = \delta[n-1] + \delta[n] + 2\delta[n+1]$, (d) $x_4[n] = \delta[-1] + \delta[0] + 2\delta[1]$, (e) $x_5[n] = \mu[n] - \mu[n-4]$, (f) $x_6[n] = x_3[-n+1]$.

Solution: There are different ways to draw discrete-time signals. Here we use "pins" or "stems", which emphasizes that the sequence is discrete-time. Compute the values for each n using sin(), $\delta[n]$, $\mu[n]$ functions, for example, in (c)

n	$\delta[n-1]$	$\delta[n]$	$2\delta[n+1]$	$x[n]$
-2	$\delta[-2-1] = 0$	$\delta[-2] = 0$	$2\delta[-2+1] = 0$	$0+0+0 = 0$
-1	$\delta[-1-1] = 0$	$\delta[-1] = 0$	$2\delta[-1+1] = 2$	$0+0+2 = 2$
0	$\delta[0-1] = 0$	$\delta[0] = 1$	$2\delta[0+1] = 0$	$0+1+0 = \underline{1}$
1	$\delta[1-1] = 1$	$\delta[1] = 0$	$2\delta[1+1] = 0$	$1+0+0 = 1$
2	$\delta[2-1] = 0$	$\delta[2] = 0$	$2\delta[2+1] = 0$	$0+0+0 = 0$

See the results in Figure 31. Note that in (b) the argument for the sine function is always 2π-multiple. In (d) there are only constants $\delta[-1] = \delta[1] = 0$ and $\delta[0] = 1$ from the definition.

The discrete-time signal is purely a sequence of numbers, e.g. in (c), $x_3[n] = \{2, \underline{1}, 1\}$, where the underlined position is at $n = 0$. Non-zero values can be also listed, e.g., $x_3[-1] = 2$, $x_3[0] = 1$, and $x_3[1] = 1$.

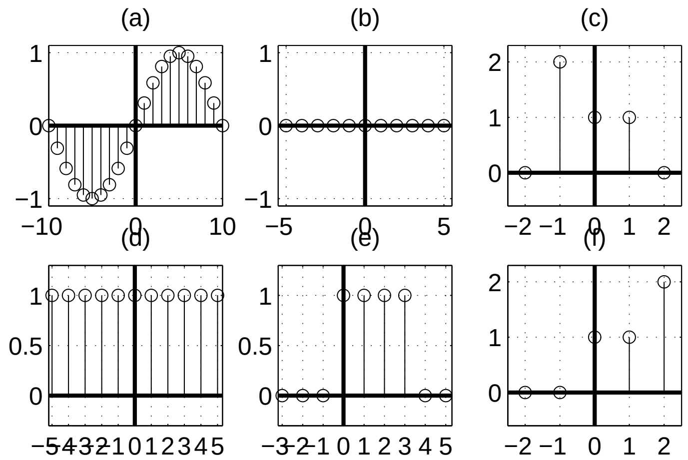


Figure 31: Sequences of Problem 12. Top row: (a)-(c), bottom: (d)-(f).

13. **Problem:** Which of the following signals are periodic? Define the length of the fundamental period and frequency for periodic signals. (a) $x(t) = 3\cos(\frac{8\pi}{31}t)$, (b) $x[n] = 3\cos(\frac{8\pi}{31}n)$, (c) $x(t) = \cos(\frac{\pi}{8}t^2)$, (d) $x[n] = 2\cos(\frac{\pi}{6}n - \pi/8) + \sin(\frac{\pi}{8}n)$, (e) $x[n] = \{\ldots, \underline{2}, 0, 1, 2, 0, 1, 2, 0, 1, \ldots\}$, (f) $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-4k] + \delta[n-4k-1]$.

    Solution: Continuous-time signal $x(t)$ is periodic, if there exists period $T \in \mathbb{R}$, for which $x(t) = x(t+T)$, $\forall t$. The fundamental period is the smallest $T_0 > 0$.

    Discrete-time signal (sequence) $x[n]$ is periodic, if there exists period $N \in \mathbb{Z}$, for which $x[n] = x[n+N]$, $\forall n \in \mathbb{Z}$. The fundamental period is the smallest $N_0 > 0$.

    The analysis is often done for sines or cosines which are 2π-periodic. Replace t by $t+T$ (n by $n+N$) and try if the equation $x(t) = x(t+T)$ holds. Note that the amplitude or phase shift does not have effect on periodicity. The exponential function $e^{j\omega}$ is also 2π-periodic.

    Another way to find the period of sine is to express the function in form of $x(t) = \sin(2\pi \cdot f \cdot t)$ where f is frequency ($\Omega = 2\pi f$ is angular frequency). Then $T = 1/f$.

    If there is a sum of cosines, like in (d), one has to find period T_0 (N_0), to which all periods of individual cosines are multiples. Correspondingly, in frequency domain one has to find a fundamental frequency f_0, with which all individual frequencies can be represented.

    a) Periodic. When $T = (31/4)k$, then the original cosine argument is added 2π-multiple, and $x(t) = x(t+T)$ holds. The fundamental period is the shortest period $T_0 = 31/4$.

    $$\begin{aligned} x(t) &= 3\cos(\frac{8\pi}{31}t) = 3\cos(\frac{8\pi}{31}(t+T)) = 3\cos(\frac{8\pi}{31}t + \frac{8\pi}{31}T) \\ &= 3\cos(\frac{8\pi}{31}t + 2\pi(\frac{4}{31}T)) \end{aligned}$$

    b) Periodic, $N_0 = (31/4)\cdot k$. The period N_0 has to be integer, so the smallest possible $k = 4$ gives the length of the fundamental period $N_0 = 31$.

    $$\begin{aligned} x[n] &= 3\cos(\frac{8\pi}{31}n) = 3\cos(\frac{8\pi}{31}(n+N)) = 3\cos(\frac{8\pi}{31}n + \frac{8\pi}{31}N) \\ &= 3\cos(\frac{8\pi}{31}n + 2\pi(\frac{4}{31}N)) \end{aligned}$$

    Notice also the difference of the results in (a) and (b), where $x(t) = x(t+(31/4))$, but $x[n] = x[n+31]$. The signals are plotted in the same axis in Figure 32.

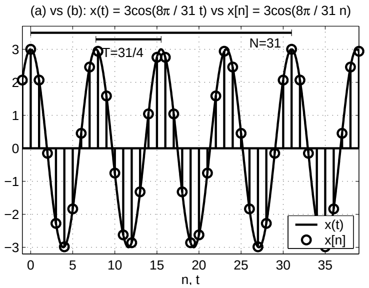


Figure 32: A visualization of the difference of fundamental periods of similar looking analog and discrete-time signals in Problem 13(a) and (b). $T_0 = 31/4$ but $N_0 = 31$.

c) Non-periodic, the latter term depends on t. The result can be also seen in Figure 33(c).

$$\begin{aligned} x(t) &= \cos(\frac{\pi}{8}t^2) = \cos(\frac{\pi}{8}(t+T)^2) = \cos(\frac{\pi}{8}t^2 + \frac{\pi}{8}(2tT + T^2)) \\ &= \cos(\frac{\pi}{8}t^2 + 2\pi(\frac{tT}{8} + \frac{T^2}{16})) \end{aligned}$$

d) Periodic, $N_0 = 48$. The fundamental period is the least common multiple (LCM) of individual periods $N_1 = 12$, $N_2 = 16 \Rightarrow N_0 = 4N_1 = 3N_2 = 48$. On the other hand the fundamental angular frequency is the greatest common divisor (GCD) of individual frequencies ($\omega = 2\pi/N$): $\omega_1 = \pi/6$, $\omega_2 = \pi/8 \Rightarrow \omega_0 = \pi/24, \Rightarrow \omega_1 = 4\omega_0$, $\omega_2 = 3\omega_0$. More about computing LCM and GCD can be found, e.g. "Beta, Mathematics Handbook for Science and Engineering". There are Matlab commands `lcm` and `gcd`, too.

$$x[n] = 2\cos(\frac{\pi}{6}n - \pi/8) + \sin(\frac{\pi}{8}n) = 2\cos(2\pi\frac{1}{12}n - \pi/8) + \sin(2\pi\frac{1}{16}n)$$

e) (Assume that) the period is $N_0 = 3$, i.e. $x[0] = x[\pm 3k] = 2$, $x[1] = x[\pm 3k + 1] = 0$, $x[2] = x[\pm 3k + 2] = 1$, where the integer $k > 0$.

f) $N_0 = 4$. "Open" the sequence if you do not see it directly:

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{+\infty} \delta[n-4k] + \delta[n-4k-1] \\ &= \ldots + \underbrace{\delta[n+4] + \delta[n+4-1]}_{k=-1} + \underbrace{\delta[0] + \delta[n-1]}_{k=0} + \underbrace{\delta[n-4] + \delta[n-4+1]}_{k=1} + \ldots \\ &= \{\ldots,\ 1,\ 1,\ 0,\ 0,\ \underline{1},\ 1,\ 0,\ 0,\ 1,\ 1,\ 0,\ 0, \ldots\} \end{aligned}$$

Periodicity of signal is often easy to see from the signal plot, see Figure 33. The signal (c) is clearly not periodic. Real-life signals (e.g. speech signal) are seldom periodic in strict sense; almost periodic signals are sometimes called "quasi-periodic".

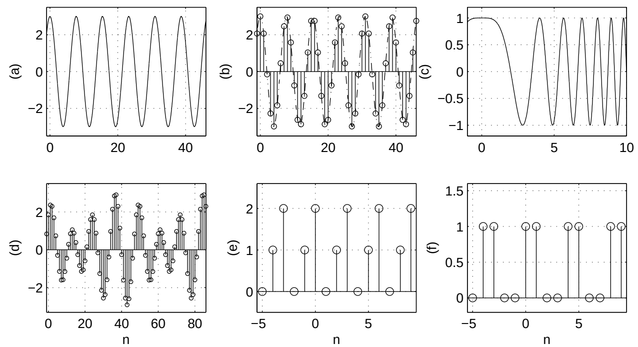


Figure 33: Signals and sequences in Problem 13, (a)..(c) in top row, (d)..(f) in bottom row. It can be seen that (at least) (c) is not periodic in the scene shown.

14. **Problem:** Express the input-output relations of the discrete-time systems in Figure 35.

Solution: In this problem there are several types of discrete-time systems. Notice that the scope of this course is LTI systems (linear and time-invariant). LTI systems are very easy to detect, they are relatively simple but very useful. In this course the system input $x[n]$ and output $y[n]$ are 1-dimensional except some examples with pictures (2D). For LTI-systems the input-output relation can be written with a difference equation or a set of difference equations.

There are some basic operations on sequences (signals) in discrete-time systems (x refers to input to the system / operation, y output) shown also in Figure 34:

- sum of signals (sequences) $y[n] = x_1[n] + x_2[n]$
- signal multiplication (by constant) $y[n] = a\,x[n]$
- delay or advance of signal $y[n] = x[n \pm k]$
- branch / pick-off node $y_1[n] = x[n]$, $y_2[n] = x[n]$
- product of signals, modulator (non-LTI systems) $y[n] = x_1[n] \cdot x_2[n]$

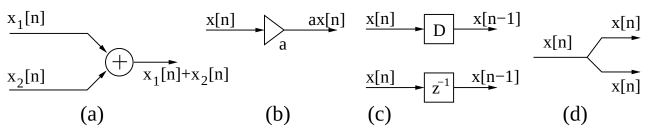


Figure 34: Problem 14: Basic operations in discrete-time systems, (a) sum of sequencies, (b) amplification by constant, (c) unit delay (D, T, or z^{-1}), and (d) branch / pick-off node.

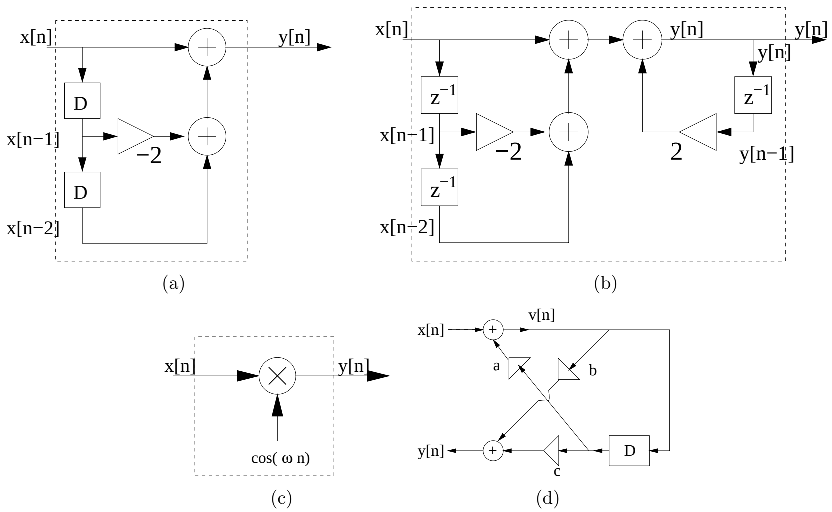


Figure 35: Problem 14: Discrete-time systems, also at page 8.

a) Difference equation: $y[n] = x[n] - 2x[n-1] + x[n-2]$. LTI-filter, type FIR (see Problem 15).

b) The memory registers / unit delays are often drawn either "D" (delay) or "z^{-1} (refers to a delay in z-transform). Note that the output is fed back in the loop. The left part of the filter is the same as (a). The sequence right after the second summing on top line is $y[n]$ which goes both to the output and down to feedback loop. Therefore the terms coming into the last summing unit are $x[n] - 2x[n-1] + x[n-2]$ from left and $2y[n-1]$ from the loop. The difference equation is

$$y[n] = 2y[n-1] + x[n] - 2x[n-1] + x[n-2]$$

The system is LTI and type IIR (see Problem 15).

c) Input signal $x[n]$ is multiplied by a sequence $\cos(\omega n)$ (not a constant). This operation is called modulation and is not LTI. The relation can be written as

$$y[n] = x[n] \cdot \cos(\omega n)$$

d) This is so called lattice structure (*Mitra 2Ed Sec. 6 / 3Ed Sec. 8*). In order to get relationship between $x[n]$ and $y[n]$ temporary variables are used after each summing unit. In this case, there is one temporary variable $v[n]$, and the set of difference equations is

$$\begin{aligned} v[n] &= x[n] + a\,v[n-1] \\ y[n] &= b\,v[n] + c\,v[n-1] \end{aligned}$$

The temporary variable $v[n]$ can be simplified away, but it is easier to determine the transfer function $H(z)$ in frequency domain and then apply inverse z-transform, which is discussed later. The system is LTI and IIR (see Problem 15).

Remark. The simplified difference equation for the system in (d) can be received by eliminating all temporary $v[n]$ sequences:

$$\begin{aligned} x[n] &= v[n] - a\,v[n-1] && \mid\; x \text{ on left side} \\ y[n] &= b\,v[n] + c\,v[n-1] && \mid\; y \text{ on left side} \\ -bx[n] &= -b\,v[n] + ab\,v[n-1] && \mid\; y[n] - bx[n] \text{ cancels } v[n] \\ -ay[n-1] &= -ab\,v[n-1] - ac\,v[n-2] && \\ -cx[n-1] &= -c\,v[n-1] + ac\,v[n-2] && \mid \text{ all } v[n-1], v[n-2] \text{ cancelled} \end{aligned}$$

which finally gives $y[n] = ay[n-1] + bx[n] + cx[n-1]$.

The discrete-time system does some computation for sequencies of numbers. Therefore it is straightforward to write down a computer program, e.g. in (a),

```
x1 := 0; x2 := 0;
while TRUE {
  x2 := x1;
  x1 := x0;
  x0 := read_next_input(input_stream);
  y  := x0 - 2*x1 + x2;
  write_output(output_stream, y);
}
```

or if all samples are known and in a vector,

```
for (k = 2; k <= length(x); k++) {
  y[k] := x[k] - 2*x[k-1] + x[k-2];
}
```

15. **Problem:** Look at the flow (block) diagrams in Figure 36.

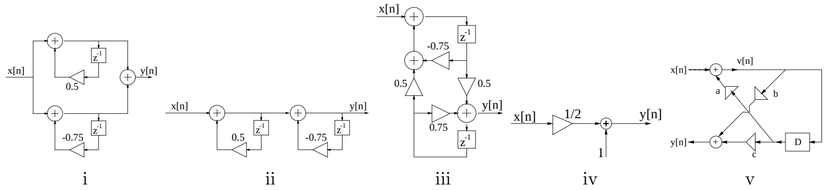


Figure 36: Flow diagrams of Problem 15, also at page 7.

a) What does LTI mean? How to prove or recognise LTI systems?
b) Which systems are linear and time-invariant (LTI)?
c) Which systems have feedback?
d) Which LTI systems are FIR and which are IIR?

Solution: In this problem we try to recognise LTI systems by their layout.

a) LTI = linear AND time-invariant (=shift-invariant) system. These two properties belong to a system not to a signal. Other system properties can be, e.g. stability, causality, or if it needs memory or if it can be inverted.
See Problem 16 for mathematical proofs.
Recognition of LTI systems from the flow (block) diagrams: there are only (1) sums of signals, (2) multiplication by a contant, (3) delays or advances. The components were introduced in Problem 14, see Figure 34 at page 39.
LTI systems can be represented with a linear constant coefficient difference (or differential in case of analog system) equation
$$\sum_k d_k\, y[n-k] = \sum_k p_k\, x[n-k]$$
where $\{d_k\}$ and $\{p_k\}$ are constants. Often in practice, we use causal finite-dimensional LTI systems $\sum_{k=0}^{N} d_k\, y[n-k] = \sum_{k=0}^{M} p_k\, x[n-k]$, where the order of the system is given by $\max\{N, M\}$ (*Mitra 2Ed Sec. 2.6.0 / 3Ed Sec. 2.7.0*). If the system cannot be written in the format above, it is not a (causal) LTI system.
b) LTI? Only summing, delays, amplifications by constants. (i) Yes, (ii) Yes, (iii) Yes, (iv) No, adding a constant, (v) Yes.
c) Feedback means that some of the output (or internal) values are fed back in the system. Computation can be said to be recursive or iterative. There are loops in (i), (ii), (iii), and (v).
d) FIR = Finite (length) Impulse Response. IIR = Infinite (length) Impulse Response.
If the system has a feedback loop somewhere in the structure, it is also IIR at the same time. The output value is computed using older output values, i.e. there is recursion. This can be seen that there are also terms $y[n-k]$, $k \neq 0$, in the difference equation.
If there is no loop and computation flows forward all the time, then the system is FIR. This can be seen that there is only the term $y[n]$ in the left side of the difference equation above.
FIR: (iv) has an impulse response of finite length but it is not LTI. IIR: (i), (ii), (iii), and (v) have infinite (length) impulse response because of feedback loops.

16. **Problem:** Determine if the system is (1) linear, (2) causal, (3) stable, and (4) shift-invariant.

a) $y[n] = x^3[n]$,
b) $y[n] = \gamma + \sum_{l=-2}^{2} x[n-l]$, γ is a nonzero constant,
c) $y[n] = \alpha x[-n]$, α is a nonzero constant.

Solution: Properties of the discrete-time system, see (*Mitra 2Ed Sec. 2.4.1, 2.5.3, 2.5.4 / 3Ed Sec. 2.4.2, 2.5.3., 2.5.4*).

Linearity:
If $y_1[n]$ and $y_2[n]$ are the responses to the input sequences $x_1[n]$ and $x_2[n]$, respectively, then for an input
$$x[n] = \alpha x_1[n] + \beta x_2[n],$$
the response is given by
$$y[n] = \alpha y_1[n] + \beta y_2[n].$$

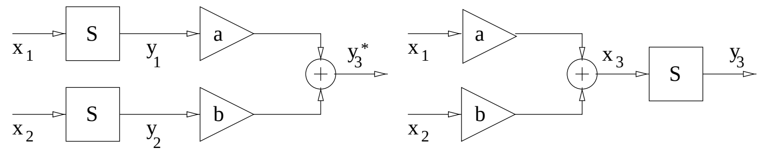


Figure 37: Linearity. If the linear combination of outputs of x_1 and x_2 is the same as the output of the linear combination of inputs, then the system is linear.

Causality:
The n_0-th output sample $y[n_0]$ depends only on previous output values and input samples $x[n]$ for $n \leq n_0$, and does not depend on input samples for $n > n_0$. In case of LTI-system, the system is causal if and only if impulse response $h[n] = 0$ for all $n < 0$.

Stability:
Bounded input, bounded output (BIBO) stability: If a bounded input (B_x is a finite constant)
$$|x[n]| < B_x < \infty, \qquad \forall n$$
produces a bounded output (B_y is a finite constant)
$$|y[n]| < B_y < \infty, \qquad \forall n$$
as a response then the system is BIBO stable (see (a) and (b) at Page 43 for details). In case of LTI-system, the system is stable if and only if $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

Time/Shift-invariance:
If $y_1[n]$ is the response to an input $x_1[n]$, then the response to an input
$$x[n] = x_1[n - n_0]$$
is simply
$$y[n] = y_1[n - n_0],$$
where n_0 is any positive or negative integer.

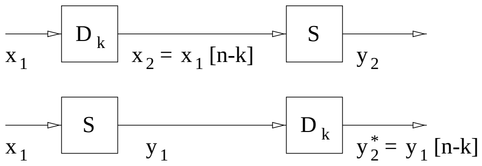


Figure 38: Time invariance. If the output of delayed input is the same as delayed output, then the system is time-invariant.

a) $y[n] = x^3[n]$.
Take inputs $x_1[n]$ and $x_2[n]$, the outputs are then $y_1[n] = x_1^3[n]$ and $y_2[n] = x_2^3[n]$. Now the linear combination of the input signals is $x_3[n] = \alpha x_1[n] + \beta x_2[n]$ and the output is

$$y_3[n] = (\alpha x_1[n] + \beta x_2[n])^3 \neq \alpha x_1^3[n] + \beta x_2^3[n] = \alpha y_1[n] + \beta y_2[n].$$

Hence the system is **not linear**.
Since there is no output before the input hence the system is **causal**.
The system is **stable**: Assume $|x[n]| < B_x$, then

$$|y[n]| = |x^3[n]| \leq |x[n]|^3 < B_x^3 = B_y < \infty.$$

The system is **time-invariant**: Assume input $x_1[n]$ and output $y_1[n]$, then response of input $x[n] = x_1[n - n_0]$ is

$$y[n] = (x[n])^3 = (x_1[n - n_0])^3 = y_1[n - n_0]$$

b) $y[n] = \gamma + \sum_{l=-2}^{2} x[n - l]$, γ is a nonzero constant.
Use linear combination $\alpha x_1[n] + \beta x_2[n]$ as the input

$$\begin{aligned} y_3[n] &= \gamma + \sum_{l=-2}^{2} (\alpha x_1[n - l] + \beta x_2[n - l]) \\ &= 0.5\gamma + \alpha \sum_{l=-2}^{2} x_1[n - l] + 0.5\gamma + \beta \sum_{l=-2}^{2} x_2[n - l] \\ &\neq \alpha\gamma + \alpha \sum_{l=-2}^{2} x_1[n - l] + \beta\gamma + \beta \sum_{l=-2}^{2} x_2[n - l] \\ &= \alpha y_1[n] + \beta y_2[n], \end{aligned}$$

where α and β are not fixed. The system is hence **nonlinear**.
The system is **not causal**, because there can be output before input, when $l \in [-2, -1]$.
System is **stable**: Assume bounded input $|x[n]| < B_x$, then

$$|y[n]| = |\gamma + \sum_{l=-2}^{2} x[n - l]| \leq |\gamma| + \sum_{l=-2}^{2} |x[n - l]| < |\gamma| + 5B_x = B_y < \infty$$

The system is also **time-invariant**: Assume input $x_1[n]$ and output $y_1[n]$, then response of input $x[n] = x_1[n - n_0]$ is

$$y[n] = \gamma + \sum_{l=-2}^{2} x_1[n - n_0] = y_1[n - n_0].$$

c) $y[n] = \alpha x[-n]$, α is a nonzero constant.
The system is **linear**, **stable** and **noncausal**.
Assume inputs $x_1[n]$,$x[n]$ and outputs $y_1[n]$, $y[n]$, respectively, then

$$\begin{aligned} y[n] &= \alpha x[-n], \\ y_1[n] &= \alpha x_1[-n]. \end{aligned}$$

Let $x[n] = x_1[n - n_0]$, then

$$\begin{aligned} y[n] &= \alpha x[-n] = \alpha x_1[-n - n_0] \\ &\neq \alpha x_1[n_0 - n] = \alpha x_1[-(n - n_0)] = y_1[n - n_0] \end{aligned}$$

and the system is **not time-invariant**.

17. **Problem:**

a) What is the impulse response of the system in Figure 39(a)? What is the connection to the difference equation? Is this LTI system stable/causal?

b) What are the first five values of impulse response of the system in Figure 39(b)?

c) What are the first five values of impulse response of the system in Figure 39(d)?

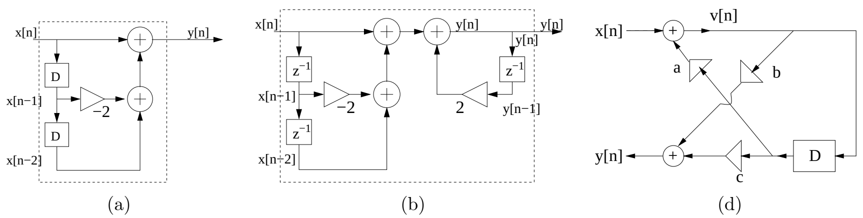


Figure 39: Discrete-time systems for Problems 17 and 18.

Solution: Impulse response $h[n]$ is the response of the system to the input $\delta[n]$. LTI discrete-time system is completely specified by its impulse response $h[n]$ (*Mitra 2Ed Sec. 2.5.1 / 3Ed Sec. 2.5.1*). For a LTI system (see Problems 15 and 16) the stability condition is $\sum_n |h[n]| < \infty$ and the causability condition $h[n] = 0, \forall n < 0$.

If the impulse response $h[n]$ is known for a LTI system, then the output $y[n]$ can be computed for any input $x[n]$ by convolution.

Impulse $\delta[n]$	$\longrightarrow$	Any discrete-time system	$\longrightarrow$	Impulse response $h_i[n]$
Input sequence $x[n]$	$\longrightarrow$	Discrete-time LTI system $h[n]$	$\longrightarrow$	Output sequence $y[n]$ $y[n] = h[n] \circledast x[n]$

a) Difference equation of the system is $y[n] = x[n] - 2x[n-1] + x[n-2]$. Let the input be $\delta[n]$ and read what comes out.

n	$x[n] = \delta[n]$	$-2x[n-1]$	$x[n-2]$	$y[n] = x[n] - 2x[n-1] + x[n-2]$
...	0	0	0	0
-1	0	0	0	0
0	1	0	0	1
1	0	-2	0	-2
2	0	0	1	1
3	0	0	0	0
...	0	0	0	0

The impulse response is

$$h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

The length $L\{.\}$ of the impulse response is finite, $L\{h[n]\} = 3 < \infty$. So, the filter is FIR (finite impulse response).

Notice that in case of FIR filter (no feedbacks, flow always going forward), the impulse response can be easily gotten from the corresponding difference equation just by replacing y by h and each x by δ (*Mitra 2Ed Sec. 2.5.1 / 3Ed Sec. 2.5.1*).

All FIR systems are always stable because the length of impulse response is finite, and therefore also the sum of absolute values is finite: $\sum |h[n]| < \infty$, in this case $\sum |h[n]| = 1 + 2 + 1 = 4 < \infty$.

This FIR system is causal while $h[n] = 0$ for all $n < 0$.

b) There is a feedback in the filter whose difference equation is

$$y[n] = 2y[n-1] + x[n] - 2x[n-1] + x[n-2]$$

The impulse response can be expressed in a closed form from the transfer function $H(z)$ by inverse z-transform (discussed later). However, the impulse reponse is the response for impulse, so just feed a delta function in and read what comes out. The initial value $y[-1]$ is by default zero.

n	$x[n] = \delta[n]$	$-2x[n-1]$	$x[n-2]$	$2y[n-1]$	$y[n] = 2y[n-1] + \ldots$
...	0	0	0	0	0
-1	0	0	0	0	0
0	1	0	0	0	1
1	0	-2	0	2	0
2	0	0	1	0	1
3	0	0	0	2	2
4	0	0	0	4	4
...	0	0	0	...	...

The system is clearly causal but it seems not to be stable while the output is not bounded. The stability of IIR systems have to be checked every time, and there will be easy tools for that later (poles of $H(z)$ outside the unit circle).

So, the first values of $h[n] = \{\underline{1},\ 0,\ 1,\ 2,\ 4, \ldots\}$ from which we can guess that the closed form equation is $h[n] = \delta[n] + 2^{n-2}\mu[n-2]$.

c) A set of difference equations can be written,

$$\begin{aligned} v[n] &= x[n] + a\,v[n-1] \\ y[n] &= b\,v[n] + c\,v[n-1] \end{aligned}$$

Just like in (b), the columns for temporary values are computed, and finally the first values of the impulse response are

$$h[n] = \{\underline{b},\ ba + c,\ ba^2 + ca,\ ba^3 + ca^2,\ ba^4 + ca^3, \ldots\}$$

from which it can be guessed that the closed form representation for the impulse response is $h[n] = ba^n\mu[n] + ca^{n-1}\mu[n-1]$.

18. **Problem:** Step response $s[n]$ is the response of the system to the input $\mu[n]$. What are the step responses of systems in Figures 39(a) and (b), see Page 45.

    Solution: Unit step response, or shortly step response $s[n]$ is the response of the system to the input $\mu[n]$ (*Mitra 2Ed Sec. 2.4.2 / 3Ed Sec. 2.4.3*). Step response can be computed easily from the impulse response $h[n]$ by cumulative sum (accumulator)

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

Now, in (a) the impulse response is $h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$, and the step response is

$$s[n] = \{\ldots, 0, 0, \underline{1}, -1, 0, 0, \ldots\}$$

which can be also seen by feeding ones to the input and reading the output. The steady-state response (*Mitra 2Ed Sec. 4.2.3 / 3Ed Sec. 3.8.5*) converges quickly to zero.

In (b) the impulse response diverges $h[n] = \delta[n] + \delta[n-2] + 2\delta[n-3] + 4\delta[n-4] + \ldots$, as well as the step response

$$s[n] = \{\ldots, 0, 0, \underline{1}, 1, 3, 7, \ldots\}$$

19. **Problem:** Compute the convolution of two signals $x_1(t)$ and $x_2(t)$ in both cases (a) and (b) in Figure 5, page 9.

    Solution: Continuous-time linear convolution of two signals $x_1(t)$ and $x_2(t)$ is defined by

$$y(t) = x_1(t) \circledast x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, \mathrm{d}\tau$$

You can see an example of graphical convolution in Java applet in URL `http://www.jhu.edu/~signals/convolve/index.html`. Sketch the signals $x_1(t)$ and $x_2(t)$ of Figure 5 into the boxes. The other signal is flipped around. When sliding the flipped signal to right over the other signal, the integral of the product is computed. At certain point t_0 the integral gives the convolution output $y(t_0)$.

In (a) the result can be seen in Figure 40(a). In (b) the arrows are impulses $\delta(t)$ which are signals having the area of unity and being infinitely narrow, i.e. the height in infinite. Convolving a signal with an impulse $\delta(t)$ can be considered as copying the signal at each place where impulse lies, see Figure 40(b).

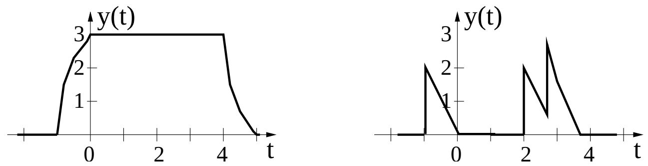


Figure 40: Problem 19: convolution results $y(t)$, left: (a), right: (b).

Remark. The continuous-time convolution contains the product of two signals and taking integral of the product. In practise, the convolution can seldom be computed in closed form. However, in (a) the signals are

$$\begin{aligned} x_1(t) &= \begin{cases} 3, & -1 \le t < 4 \\ 0, & \text{elsewhere} \end{cases} \\ x_2(t) &= \begin{cases} 2-2t, & 0 \le t < 1 \\ 0, & \text{elsewhere} \end{cases} \end{aligned}$$

The flipped signal is $x_2(t-\tau) = 2 - 2t + 2\tau$, and the convolution integral is $y(t) = \int_\tau x_1(\tau) \cdot x_2(t-\tau)\, \mathrm{d}\tau$. The convolution can be computed in five cases when sliding x_2 from left to right: (1) $t < -1$, product of signals is zero, (2) $-1 < t < 0$, x_2 "penetrating", (3) $0 < t < 4$, "stable" case, (4) $4 < t < 5$, x_2 "leaving", (5) $t > 5$, again zero.

$$\begin{aligned} y(t)_{(1)} &= \int_{-\infty}^{-1} 0 \cdot (2-2t+2\tau)\, \mathrm{d}\tau = 0, \quad t < -1 \\ y(t)_{(2)} &= \int_{-1}^{t} 3 \cdot (2-2t+2\tau)\, \mathrm{d}\tau = 3 - 3t^2, \quad -1 \le t < 0 \\ y(t)_{(3)} &= \int_{t-1}^{t} 3 \cdot (2-2t+2\tau)\, \mathrm{d}\tau = 3, \quad 0 \le t < 4 \\ y(t)_{(4)} &= \int_{t-1}^{4} 3 \cdot (2-2t+2\tau)\, \mathrm{d}\tau = 3t^2 - 30t + 75, \quad 4 \le t < 5 \\ y(t)_{(5)} &= \int_{5}^{\infty} 0 \cdot (2-2t+2\tau)\, \mathrm{d}\tau = 0, \quad t \ge 5 \end{aligned}$$

20. **Problem:**

a) Compute $x[n] \circledast h[n]$, when
$x[n] = \delta[n] + \delta[n-1]$, and $h[n] = \delta[n] + \delta[n-1]$. What is the length?

b) Compute $x_1[n] \circledast x_2[n]$, when
$x_1[n] = \delta[n] + 5\delta[n-1]$, and $x_2[n] = -\delta[n-1] + 2\delta[n-2] - \delta[n-3] - 5\delta[n-4]$.
What is the length? Where does the sequence start?

c) Compute $h[n] \circledast x[n]$, when
$h[n] = 0.5^n\mu[n]$, and $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$. What is the length?

Solution: **Discrete-time** linear convolution of two sequences $h[n]$ and $x[n]$:

$$y[n] = h[n] \circledast x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The convolution is an operation for two sequences (*Mitra 2Ed Sec. 2.5.1 / 3Ed Sec. 2.5.1, 2.5.2*). There are several ways to get the convolution result. First, in (a) the convolution is considered as filtering, the other sequence is the input and the other is the inpulse response of the system, and the convolution result is the output of the system. Second, in (b) a graphical way of inverting and sliding the sequences over each other is represented. In (c) the convolution is considered as a sum of shifted and scaled sequences, "tabular method" in (*Mitra 3Ed Sec. 2.5.2*). However, even if three ways are introduced separately, they all rely on the same (and simple) definition of the convolution.

When computing discrete-time convolution $y[n] = x[n] \circledast h[n]$, it is nice know a couple of rules. Let $L\{.\}$ be a length of a sequence, e.g. $x[n] = \{3, \underline{2}, 0, 5, -2\}$, then $L\{x[n]\} = 5$.

Because LTI-system is shift-invariant, the starting point of the convolution result can be determined as a sum of starting points of the convolved sequencies. Let $A\{.\}$ be an index number of the first non-zero element, e.g. esim. $A\{x[n]\} = -1$.

It is easily seen that for the convolution result $y[n]$ it holds

$$\begin{aligned} L\{y[n]\} &= L\{x[n]\} + L\{h[n]\} - 1 \\ A\{y[n]\} &= A\{x[n]\} + A\{h[n]\} \end{aligned}$$

There are also some nice convolution demos in Internet, e.g. `http://www.jhu.edu/~signals/discreteconv2/index.html`.

a) Consider convolution as filtering with the input sequence $x[n] = \delta[n] + \delta[n-1] = \{\underline{1},\ 1\}$, and the impulse response $h[n] = \delta[n] + \delta[n-1] = \{\underline{1},\ 1\}$, of the system. The corresponding difference equation is $y[n] = x[n] + x[n-1]$, that is, the output is just the sum of the present and previous value in the input. (You can draw the flow (block) diagram for the system and verify the computation.)

n	$x[n] = \delta[n] + \delta[n-1]$	$x[n-1]$	$y[n] = x[n] + x[n-1]$
-1	0	0	$0+0=0$
0	1	0	$1+0=1$
1	1	1	$1+1=2$
2	0	1	$0+1=1$
3	0	0	$0+0=0$
4	0	0	$0+0=0$

So, the result is $x[n] \circledast h[n] = \{\underline{1},\ 2,\ 1\} = \delta[n] + 2\delta[n-1] + \delta[n-2]$, and the length is $L\{y[n]\} = 3$. The starting point can be checked: $A\{y[n]\} = A\{x[n]\} + A\{h[n]\} = 0$.

b) Another way (on-line) is computing output values at each time moment n. Graphically this means **inverting** (flipping around) the other sequence, **sliding** it over the other, and computing the output value as a dot sum. This is also illustrated with figures in (*Mitra 2Ed Ex. 2.24, p. 73-75 / 3Ed Ex. 2.26, p. 80-83*).

Now when $x_1[n] = \delta[n] + 5\delta[n-1]$ and $x_2[n] = -\delta[n-1] + 2\delta[n-2] - \delta[n-3] - 5\delta[n-4]$, then $L\{x[n]\} = 2 + 4 - 1 = 5$ and $A\{x[n]\} = 0 + 1 = 1$. Therefore we know that the convolution result is of form $x[n] = a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4] + a_5\delta[n-5]$.

$$\begin{aligned}
n = 1: \quad x[1] &= \sum_{k=-\infty}^{\infty} x_1[k]x_2[1-k] \\
&= 0 + (\underbrace{x_1[0]}_{1} \cdot \underbrace{x_2[1-0]}_{-1}) + (\underbrace{x_1[1]}_{5} \cdot \underbrace{x_2[1-1]}_{0}) + 0 \\
&= -1 \\
n = 2: \quad x[2] &= \sum_{k=-\infty}^{\infty} x_1[k]x_2[2-k] \\
&= 0 + (x_1[0] \cdot x_2[2-0]) + (x_1[1] \cdot x_2[2-1]) + 0 \\
&= 2 + (-5) = -3 \\
n = 3: \quad x[3] &= \sum_{k=-\infty}^{\infty} x_1[k]x_2[3-k] \\
&= 0 + (x_1[0] \cdot x_2[3-0]) + (x_1[1] \cdot x_2[3-1]) + 0 \\
&= -1 + 10 = 9 \\
n = 4: \quad x[4] &= -5 + (-5) = -10 \\
n = 5: \quad x[5] &= -25
\end{aligned}$$

The procedure is represented stepwise, and step $n = 3$ is shown also in Figure 41. In the top line of the figure there is the sequence $x_1[k] = \{\ldots, 0, \underline{1}, 5, 0, \ldots\}$, in the second line the shifted and inverted sequence $x_2[n-k]$. It slides from left to right when n increases, and at $n = 3$ it is $x_2[3-k] = \{\ldots, 0, -5, \underline{-1}, 2, -1, 0, \ldots\}$. The point-wise product of sequences in top rows is shown in the third line:
$\{x_1[k]x_2[3-k]\} = \{\ldots, 0, 0 \cdot (-5), \underline{1 \cdot (-1)}, 5 \cdot 2, 0 \cdot (-1), 0, \ldots\} = \{\underline{-1},\ 10\}$.
The convolved value $x[3]$ is the sum of values in the third row:
$x[3] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[3-k] = -1 + 10 = 9$.
In the bottom line there is the result for $n \leq 3$, and $n = 3$ underlined, and results for $n > 3$ are to be computed.

c) The convolution can be computed **as a sum of shifted and scaled sequences**. Now, $h[n] = 0.5^n\mu[n]$, and $x[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$, in other words $x[0] = 1$, $x[1] = 2$, $x[2] = -1$, and $x[n] = 0$, elsewhere. The division into three parts on third line emphasizes the fact that a scalar $x[k]$ is zero with all values of k except $k = \{0,\ 1,\ 2\}$.

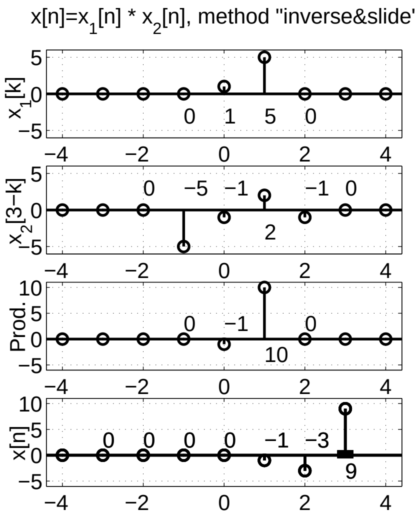


Figure 41: Problem 20(b): Linear convolution using "invert and slide". Caption from the step $n = 3$, i.e. computing the value $x[3] = 9$. See the text for more details. There is a demo Matlab program `linconv.m` to demonstrate the computation in the course web pages.

$$
\begin{aligned}
y[n] &= x[n] \circledast h[n] \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{-1} x[k]h[n-k] + \sum_{k=0}^{2} x[k]h[n-k] + \sum_{k=3}^{\infty} x[k]h[n-k] \\
&= 0 + \sum_{k=0}^{2} x[k]h[n-k] + 0 \\
&= \underbrace{x[0]}_{\text{scaling}} \cdot \underbrace{h[n-0]}_{\text{shifted seq.}} + x[1]h[n-1] + x[2]h[n-2] \\
&= 1 \cdot h[n] + 2 \cdot h[n-1] - 1 \cdot h[n-2] \\
&= 0.5^n \mu[n] + 2 \cdot 0.5^{n-1} \mu[n-1] - 0.5^{n-2} \mu[n-2] \\
&= \delta[n] + 2.5\delta[n-1] + 0.5^n \mu[n-2] \qquad | \text{ alternative}
\end{aligned}
$$

It can be seen that values of $x[n]$ were scaling factors and sequence $h[n]$ was shifted each time. While convolution is commutative ($x_1[n] \circledast x_2[n] = x_2[n] \circledast x_1[n]$), one can compute the same using values of $h[n]$ as scaling factors and shifting $x[n]$. The procedure is depicted in Figure 42. While the length of the other sequence is infinitive, so is also the length of the convolution.

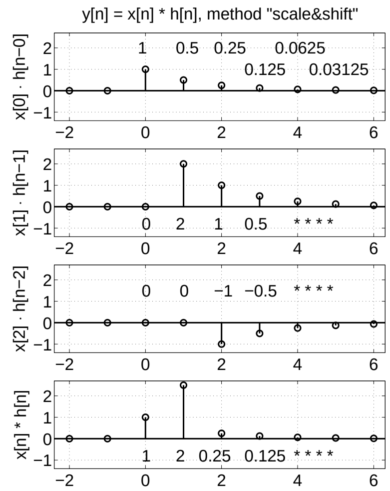


Figure 42: Problem 20(c): Linear convolution using "scaled and shifted sequences". Top line: $x[0] \cdot h[n-0] = 0.5^n \mu[n]$, second: $x[1] \cdot h[n-1] = 2 \cdot 0.5^{n-1} \mu[n-1]$, third: $x[2] \cdot h[n-2] = -1 \cdot 0.5^{n-2} \mu[n-2]$, bottom: convolution result, sum of sequences above.

21. **Problem:** Consider a LTI-system with impulse response $h[n] = \delta[n-1] - \delta[n-2]$ and input sequence $x[n] = 2\delta[n] + 3\delta[n-2]$.

    a) What is the length of convolution of $h[n]$ and $x[n]$ (without computing convolution itself)? Which index n is the first one having a non-zero item?
    b) Compute convolution $y[n] = h[n] \circledast x[n]$
    c) Consider polynomials $S(x) = 2 + 3x^2$ and $T(x) = x - x^2$. Compute the product $U(x) = S(x) \cdot T(x)$
    d) Check the result by computing the polynomial division $T(x) = U(x)/S(x)$.

Solution: An important rule of thumb for finding length $L\{.\}$ of the linear convolution (different from circular convolution):

$$y[n] = h[n] \circledast x[n] \quad \rightarrow \quad L\{y[n]\} = L\{h[n]\} + L\{x[n]\} - 1$$

The first non-zero item $A\{.\}$ for finite sequences:

$$y[n] = h[n] \circledast x[n] \quad \rightarrow \quad A\{y[n]\} = A\{h[n]\} + A\{x[n]\}$$

a) $L\{h[n]\} = 2,\ L\{x[n]\} = 3 \ \rightarrow L\{y[n]\} = 4$. Because $h[n]$ is delayed by one ($d_h = +1$) and $x[n]$ starts from the origo ($d_x = 0$), also their convolution is delayed by one: $A\{h[n]\} = 1,\ A\{x[n]\} = 0 \ \rightarrow A\{y[n]\} = 1$.
Now we know that the result is of form:

$$y[n] = a_1\delta[n-1] + a_2\delta[n-2] + a_3\delta[n-3] + a_4\delta[n-4]$$

b) Using values of $h[n] = \delta[n-1] - \delta[n-2]$ as scaling factors

$$\begin{aligned} y[n] &= h[n] \circledast x[n] \\ &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\ &= \sum_{k=1}^{2} h[k]x[n-k] \\ &= 1\cdot(2\delta[n-1] + 3\delta[n-3]) - 1\cdot(2\delta[n-2] + 3\delta[n-4]) \\ &= 2\delta[n-1] - 2\delta[n-2] + 3\delta[n-3] - 3\delta[n-4] \end{aligned}$$

c) $U(x) = S(x)\cdot T(x) = (2+3x^2)\cdot(x-x^2) = 2x - 2x^2 + 3x^3 - 3x^4$. Notice the correspondence with the result of (b), the delay is the power of x (z^{-1} in Z-transform).

d) Using long division (*Mitra 2Ed Ex. 3.35 / 3Ed Ex. 6.19*). The polynomials are $U(x) = 2x - 2x^2 + 3x^3 - 3x^4$ and $S(x) = 2 + 3x^2$,

$$\begin{array}{r|rrrr} & -x^2 & +x & & \\ 3x^2+2 & -3x^4+3x^3 & -2x^2 & +2x & \\ & 3x^4 & +2x^2 & & \\ \hline & 3x^3 & & +2x & \\ & -3x^3 & & -2x & \\ \hline & & & 0 & \end{array}$$

We get the result $x - x^2$ as expected ($h[n] = \delta[n-1] - \delta[n-2]$). Convolution and deconvolution operations can be computed using products and divisions of polynomials.

22. **Problem:** The impulse response $h_1[n]$ of a LTI system is known to be $h_1[n] = \mu[n] - \mu[n-2]$. It is connected in cascade (series) with another LTI system h_2 as in Figure 6 at page 10:

$$h_1[n] \to h_2[n] \to h_1[n]$$

Compute the impulse response $h_2[n]$, when it is known that the impulse response $h[n]$ of the whole system is $h[n] = \{\underline{1}, 5, 9, 7, 2\}$ (Table 2 on page 10).

Solution: There are three subsystems connected in cascade (series). They are all linear and time-invariant (LTI). The overall impulse response of the whole system is therefore

$$\begin{aligned} h[n] &= (h_1[n] \circledast h_2[n]) \circledast h_1[n] \\ h[n] &= (h_1[n] \circledast h_1[n]) \circledast h_2[n] \\ &= \delta[n] + 5\delta[n-1] + 9\delta[n-2] + 7\delta[n-3] + 2\delta[n-4] \end{aligned}$$

Notice that $h[n]$ and $h_1[n]$ are known but $h_2[n]$ is unknown. If one of the signals to be convolved is unknown and the convolution result is known, the operation to find the unknown is called deconvolution, inverse operation of convolution. The procedure of deconvolution is basically the same as that with convolution. If polynomial products are used, then the operation is polynomial division $H_2(x) = H(x)/(H_1(x)H_1(x))$.

First, compute $h_{11}[n] = h_1[n] \circledast h_1[n]$, with $h_1[n] = \delta[n] + \delta[n-1]$,

$$\begin{aligned} h_{11}[n] &= h_1[n] \circledast h_1[n] \\ &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

Second, compute the length (here $L\{.\}$) of $h_2[n]$. While $L\{h[n]\} = 5$, $L\{h_{11}[n]\} = 3$, and $L\{h[n]\} = L\{h_{11}[n]\} + L\{h_2[n]\} - 1$, the result is $L\{h_2[n]\} = 3$.

The index of the first non-zero element (here $A\{.\}$) is $A\{h_2[n]\} = A\{h[n]\} - A\{h_{11}[n]\} = 0 - 0 = 0$. Therefore the unknown sequence can be written as $h_2[n] = a\delta[n] + b\delta[n-1] + c\delta[n-2]$.

Third, compute the convolution, and solve the the unknown constants a, b, c.

$$\begin{aligned} h[n] &= h_{11}[n] \circledast h_2[n] \\ &= \sum_{k=-\infty}^{+\infty} h_{11}[k]h_2[n-k] = \sum_{k=0}^{2} h_{11}[k]h_2[n-k] \\ &= h_2[n-0] + 2h_2[n-1] + h_2[n-2] \\ &= (a\delta[n] + b\delta[n-1] + c\delta[n-2]) \\ &\quad +2(a\delta[n-1] + b\delta[n-2] + c\delta[n-3]) \\ &\quad +(a\delta[n-2] + b\delta[n-3] + c\delta[n-4]) \\ &= a\delta[n] + (b+2a)\delta[n-1] + (c+2b+a)\delta[n-2] + (2c+b)\delta[n-3] + c\delta[n-4] \\ &= \delta[n] + 5\delta[n-1] + 9\delta[n-2] + 7\delta[n-3] + 2\delta[n-4] \qquad | \ h[n] \text{ is known} \end{aligned}$$

The comparison between the last two lines from left gives $a = 1$, then $(b + 2\cdot 1) = 5 \Rightarrow b = 3$, then $(c + 2\cdot 3 + 1) = 9 \Rightarrow c = 2$, and also the rest values hold. In the end, the result is

$$h_2[n] = \delta[n] + 3\delta[n-1] + 2\delta[n-2]$$

which can be ensured by convolution.

23. **Problem:** Determine the expression for the impulse response of each of the LTI systems shown in Figure 43.

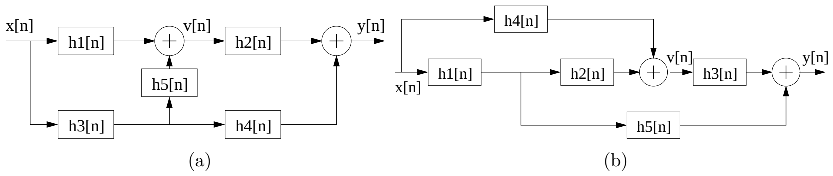


Figure 43: LTI systems with variables $x[n], v[n], y[n]$ in Problem 23.

Solution: All subsystems are LTI. Therefore we can use sum of impulse responses for parallel systems and convolution of impulse responses for cascade systems (*Mitra 2Ed Ex. 2.27 / 3Ed Ex. 2.35*).

If any temporary variables are needed, they are probably best situated right after the summing units.

a) We can derive the impulse response $h[n]$ of the whole system directly, or using a temporary variable $v[n]$ (easier!?) shown in Figure 43. The useful position for $v[n]$ is after summation.

$$\begin{aligned} v[n] &= (h_1[n] \circledast x[n]) + ((h_3[n] \circledast h_5[n]) \circledast x[n]) \\ y[n] &= (h_2[n] \circledast v[n]) + ((h_3[n] \circledast h_4[n]) \circledast x[n]) \\ &= \big((h_2[n] \circledast h_1[n]) + (h_2[n] \circledast h_3[n] \circledast h_5[n]) + (h_3[n] \circledast h_4[n])\big) \circledast x[n] \\ h[n] &= (h_2[n] \circledast h_1[n]) + (h_2[n] \circledast h_3[n] \circledast h_5[n]) + (h_3[n] \circledast h_4[n]) \end{aligned}$$

b) In the same way as in (a).

$$\begin{aligned} v[n] &= (h_4[n] \circledast x[n]) + ((h_1[n] \circledast h_2[n]) \circledast x[n]) \\ y[n] &= (h_3[n] \circledast v[n]) + ((h_1[n] \circledast h_5[n]) \circledast x[n]) \\ &= \big((h_3[n] \circledast h_4[n]) + (h_1[n] \circledast h_2[n] \circledast h_3[n]) + (h_1[n] \circledast h_5[n])\big) \circledast x[n] \\ h[n] &= (h_3[n] \circledast h_4[n]) + (h_1[n] \circledast h_2[n] \circledast h_3[n]) + (h_1[n] \circledast h_5[n]) \end{aligned}$$

24. **Problem:** The impulse response of a digital matched filter, $h[n]$, is the time-reversed replica of the signal to be detected. The time-shift is needed in order to get a causal filter.

The (binary) signal to be detected is given by $s[n] = \{1, 1, 1, -1, -1, 1, -1\}$. Consider an input sequence $x[n]$ which is a periodic sequence repeating $s[n]$. Determine $h[n]$ and the result of filtering $y[n] = h[n] \circledast x[n]$.

Solution: Matched filter. Let $s[n]$ be a (binary) 7-bit long codeword to be detected, $x[n]$ an input signal of repeated $s[n]$, and the impulse response of the matched filter $h[n] = s[-n]$:

$$\begin{aligned} s[n] &= \{1, 1, 1, -1, -1, 1, -1\} \\ x[n] &= \{\ldots, s[n], s[n], s[n], \ldots\} = \\ &= \{\ldots, 1, 1, 1, -1, -1, 1, -1, 1, 1, 1, -1, -1, 1, -1, 1, 1, 1, -1, -1, 1, -1, \ldots\} \\ h[n] &= s[-n] = \{-1, 1, -1, -1, 1, 1, 1\} \end{aligned}$$

The convolution result $y[n] = h[n] \circledast x[n]$ is shown in Figure 44.

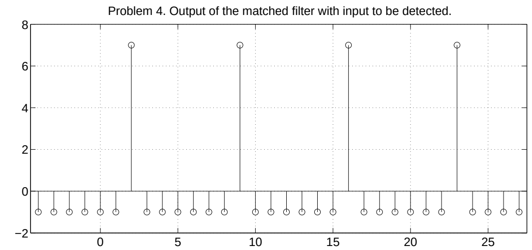


Figure 44: Convolution result of the matched filter and desired sequence in Problem 24.

The signal $s[n]$ was chosen so, that the every seventh sample (length of $s[n]$) in output is high, and all others are low. If the signal $s[n]$ were different, there would be not so clear peaks in the convolution result.

Note that convolution and cross-correlation have a close connection (*Mitra 2Ed Eq. 2.106, p. 89 / 3Ed Eq. 2.127, p. 101*)

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} y[n]x[-(l-n)] = y[l] \circledast x[-l]$$

25. **Problem:** Determine the autocorrelation sequence of the sequence

$$x_1[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

and show that it is an even sequence. What is the location of the maximum value of the autocorrelation sequence?

Solution: Cross-correlation sequence $r_{xy}[l]$ of two sequences and autocorrelation sequence $r_{xx}[l]$ with lag $l = 0, \pm 1, \pm 2, \ldots$ are defined (*Mitra 2Ed Sec. 2.7 / 3Ed Sec. 2.9*)

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]y[n-l] \qquad\qquad r_{xx}[l] = \sum_{n=-\infty}^{\infty} x[n]x[n-l]$$

$$\begin{aligned} r_{xx}[l] &= \sum_{n=-\infty}^{\infty} x_1[n]x_1[n-l] \\ &= \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] \alpha^{n-l} \mu[n-l] \\ &= \sum_{n=0}^{\infty} \alpha^{2n-l} \mu[n-l] \\ &= \begin{cases} \sum_{n=0}^{\infty} \alpha^{2n-l} = \frac{\alpha^{-l}}{1-\alpha^2}, & \text{for } l < 0 \\ \sum_{n=l}^{\infty} \alpha^{2n-l} = \frac{\alpha^{-l}}{1-\alpha^2} - \frac{\alpha^{-l}-\alpha^{l}}{1-\alpha^2} = \frac{\alpha^{l}}{1-\alpha^2}, & \text{for } l \geq 0 \end{cases} \end{aligned}$$

Note for the lag $l \geq 0$, $r_{xx}[l] = \frac{\alpha^l}{1-\alpha^2}$, and for $l < 0$, $r_{xx}[l] = \frac{\alpha^{-l}}{1-\alpha^2}$.

Replacing l with $-l$ in the second expression we get $r_{xx}[-l] = \frac{\alpha^{-(-l)}}{1-\alpha^2} = r_{xx}[l]$.

Hence, $r_{xx}[l]$ is an even function of l.

Maximum value of $r_{xx}[l]$ occurs at $l = 0$ since α^l is a decaying function for increasing l when $|\alpha| < 1$.

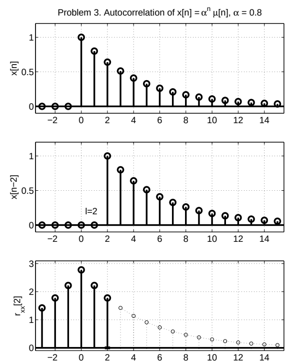


Figure 45: Autocorrelation sequence in Problem 25. Top: $x[n]$, middle: $x[n-2]$, bottom: $r_{xx}[l], r_{xx}[2] = \sum_k x[k]x[k-2]$.

26. **Problem:** Sketch the following signals in time-domain and their (amplitude) spectra in frequency-domain.

a) $x_1(t) = \cos(2\pi\, 500\, t)$

b) $x_2(t) = 4\cos(2\pi\, 200\, t) + 2\sin(2\pi\, 300\, t)$

c) $x_3(t) = e^{-j(2\pi\, 250 t)} + e^{j(2\pi\, 250 t)}$

d) $x_4(t) = x_1(t) + x_2(t) + x_3(t)$

Solution: Fourier series / Continuous-time Fourier transform (CFT or CTFT) / discrete-time Fourier transform (DTFT) decomposes the signal to its frequency components. Cosine and exponential function have a close relationship via Euler's formula:

$$\cos(\Omega t) = 0.5 \cdot (e^{j\Omega t} + e^{-j\Omega t})$$

Ideally, each real cosine component $x_i(t) = A_i \cos(2\pi f_i t + \theta_i)$ is a peak at frequency f_i in an one-side spectrum or a peak pair at frequencies $-f_i$ and f_i in a two-side spectrum. So, if the signal $x(t)$ ($x[n]$) is real-valued, then the two-side spectrum $|X(j\Omega)|$ ($|X(e^{j\omega})|$) is symmetric.

The amplitude A_i expresses how strong the cosine component is.

a) A pure cosine at 500 Hz. Figure 46(a).

b) A sum of two cosines. Peaks at 200 and 300 Hz. Figure 46(b).

c) Two complex exponentials with the same amplitude and opposite frequencies can be combined to a cosine using Euler's formula. A peak at 250 Hz. Figure 46(c).

d) The sum signal contains all components in time domain as well as in frequency domain. Figure 46(d).

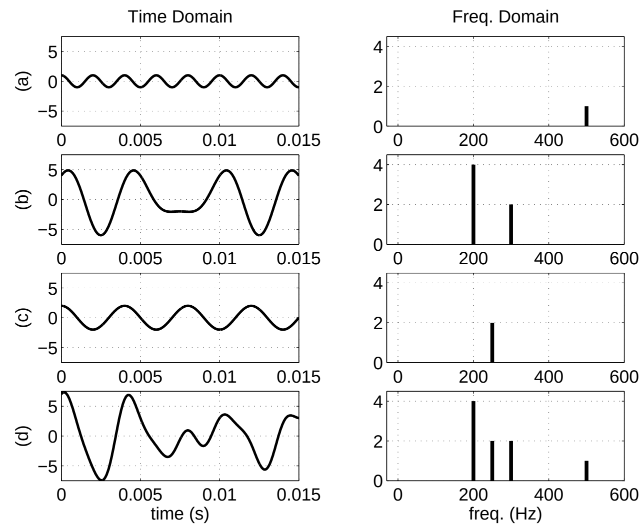


Figure 46: Signals and their one-side spectra (CFT) in Problem 26.

Remark. Typically, when computing spectra numerically ($x[n]$ instead of $x(t)$) with computer, the peaks "spread". There is the discrete Fourier transform (DFT) of signal $x_4[n] \leftarrow x_4(t)$ in Figure 47, DFT using N=40 points in (a), and DFT using N=41 points in (b), and both having the sampling frequency $f_s = 2000$ Hz. So, in (a) the resolution f_0 of the frequency is exactly 50 Hz, whereas in (b) it is 2000 Hz /41 = 48.78 Hz. The components of the signal are multiples of 50 Hz ($4 \cdot 50 = 200$, etc.) but not multiples of 48.78 Hz. This example is executed using the command `fft` in Matlab.

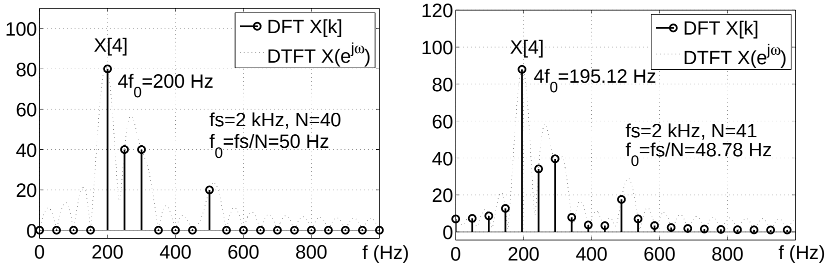


Figure 47: Discrete Fourier Transform (DFT) of the same signal (now discrete-time $x_4[n]$) as in Figure 46(d): (a) signal components (200, 250, 300, 500 Hz) are multiples of the frequency resolution $f_0 = 2000$ Hz /40, (b) signal components are not any more multiples of $f_0 = 2000$ Hz /41. Actually there are only four frequency components in the signal, but this cannot be observed in (b). Fourier component $X_4[4]$ is highlighted in both figures. In (a) its frequency is $4f_0 = 200$ Hz, while in (b) it is $4f_0 \approx 195$ Hz. Dashed line is the result of discrete-time Fourier transform (DTFT) where the frequency is continuous-valued. Example in Problem 26.

When analyzing spectra in any commercial software, the sequence is first "cut" with a window (Hamming, Hanning, Blackman, etc.). Windows and their effect on spectra are discussed later in FIR filter design.

27. **Problem:** Suppose that a real sequence $x[n]$ and its discrete-time Fourier transform (DTFT) $X(e^{j\omega})$ are known. The sampling frequency is f_s. At angular frequency $\omega_c = \pi/4$: $X(e^{j(\pi/4)}) = 3 + 4j$. Determine

a) $|X(e^{j(\pi/4)})|$ b) $\angle X(e^{j(\pi/4)})$
c) $X(e^{j(-\pi/4)})$ d) $X(e^{j(\pi/4+2\pi)})$
e) If $f_s = 4000$ Hz, what is f_c

Solution: Discrete-time Fourier transform (DTFT) is always 2π-periodic:

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \underbrace{e^{-j2\pi n}}_{=1} = X(e^{j\omega})$$

Complex-valued DTFT can be considered in polar coordinates

$$\begin{aligned} X(e^{j\omega}) &= |X(e^{j\omega})| \cdot e^{j\angle X(e^{j\omega})} \\ z &= r \cdot e^{j\theta} \end{aligned}$$

where $|X(e^{j\omega})|$ is (amplitude) spectrum and $\angle X(e^{j\omega})$ phase spectrum.

The value of DTFT was given at $\omega_c = \pi/4$: $X(e^{j(\pi/4)}) = 3 + 4j$.

a) $|X(e^{j(\pi/4)})| = 5$
b) $\angle X(e^{j(\pi/4)}) = \arctan(4/3) \approx 0.927$
c) $X(e^{j(-\pi/4)}) = 3 - 4j$
d) $X(e^{j(\pi/4+2\pi)}) = 3 + 4j$
e) Angular sampling frequency is $\omega_s = 2\pi$. The interesting frequency can be obtained from the ratio $(\omega_c/\omega_s) = (f_c/f_s)$. If the sampling frequency $f_s = 4000$ Hz, then

$$f_c = \frac{4000 \text{ Hz } \cdot (\pi/4)}{2\pi} = 500 \text{ Hz}.$$

28. **Problem:** Compute DTFT for each of the following sequences using the definition.

a) $x_1[n] = \delta[n-2]$
b) $x_2[n] = 0.5^n \mu[n]$
c) $x_3[n] = a[n] \cdot \cos(\frac{\pi}{5}n)$

Solution: Transforms of the sequences can be also read "directly" from the transform table. However, this time transforms are computed from the definition.

Discrete-time Fourier transform (DTFT) of sequence $x[n]$ is defined

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

a) $x_1[n] = \delta[n-2]$

$$\begin{aligned} X_1(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_1[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta[n-2]e^{-j\omega n} \\ &= e^{-j2\omega} \end{aligned}$$

b) $x_2[n] = 0.5^n \mu[n]$

$$\begin{aligned} X_2(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_2[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} 0.5^n \mu[n]e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} (0.5 \cdot e^{-j\omega})^n \\ &= \frac{1}{1 - 0.5 \cdot e^{-j\omega}} \end{aligned}$$

c) $x_3[n] = a[n]\cos(\frac{\pi}{5}n)$

$$\begin{aligned} X_3(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_3[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a[n]\cos(\frac{\pi}{5}n)e^{-j\omega n} \\ &= 0.5\sum_{n=-\infty}^{\infty} a[n]\big(e^{j\frac{\pi}{5}n} + e^{-j\frac{\pi}{5}n}\big)e^{-j\omega n} \\ &= 0.5\sum_{n=-\infty}^{\infty} a[n]\big(e^{-j(\omega-\frac{\pi}{5})n} + e^{-j(\omega+\frac{\pi}{5})n}\big) \\ &= 0.5\big(A(e^{j(\omega-\frac{\pi}{5})}) + A(e^{j(\omega+\frac{\pi}{5})})\big) \end{aligned}$$

where $A(e^{j\omega})$ is DTFT of $a[n]$. Signal $a[n]$ is modulated with $\omega = \pi/5$. In the frequency domain the spectrum $A(e^{j\omega})$ is "copied" (and scaled) at negative and positive angular frequency $\omega = \pi/5$.

The same can be seen from the Fourier transform properties. In case of modulation (product) $x_3[n] = a[n] \cdot b[n] \leftrightarrow X_3(e^{j\omega}) = \frac{1}{2\pi}A(e^{j\omega}) \circledast B(e^{j\omega})$. Now $F\{\cos(\pi n/5)\} = \pi\sum_l(\delta(\omega - \pi/5 + 2\pi l) + \delta(\omega + \pi/5 + 2\pi l))$, and the result is received.

29. **Problem:** The exponent term in DFT/IDFT is commonly written $W_N = e^{-j2\pi/N}$.

a) Compute and draw in complex plane values of W_3^k
b) Compute 3-DFT for the sequence $x[n] = \{\underline{1},\ 3,\ 2\}$.

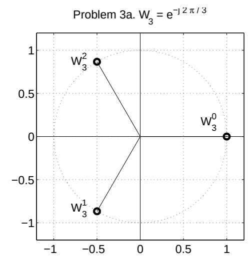
Solution: Discrete Fourier transform (DFT), left, and Inverse Fourier transform (IDFT), right, using N points are defined

$$X[k] = \sum_{n=0}^{N-1} x[n]W_N^{kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn} \qquad , W_N = e^{-\frac{j2\pi}{N}}$$

a) $W_N = e^{-\frac{j2\pi}{N}}$, now $N = 3$.

$$\begin{aligned} W_3^0 &= e^{-\frac{j2\pi}{3}\cdot 0} = 1 \\ W_3^1 &= e^{-\frac{j2\pi}{3}\cdot 1} = -0.5 - \frac{\sqrt{3}}{2}j \\ W_3^2 &= e^{-\frac{j2\pi}{3}\cdot 2} = -0.5 + \frac{\sqrt{3}}{2}j \end{aligned}$$

Notice that the exponent in W defines the angle jump in clockwise. What are the values of W_3^{kn}, when $k = 0\ldots 2$ and $n = 0\ldots 2$?. For example, $k = 1, n = 2$, we get $W_3^{1\cdot 2} = W_3^2$. Specially, $W_3^{2\cdot 2} = W_3^4 = e^{-\frac{j2\pi}{3}\cdot 4} = \underbrace{e^{-\frac{j2\pi}{3}\cdot 3}}_{=1} \cdot e^{-\frac{j2\pi}{3}\cdot 1} = W_3^1$.



k, n	0	1	2
0	W_3^0	W_3^0	W_3^0
1	W_3^0	W_3^1	W_3^2
2	W_3^0	W_3^2	W_3^1

b) The sequence $x[n] = \{\underline{1}, 3, 2\}$ is of length 3.

$$\begin{aligned} X[0] &= \sum_{n=0}^{2} x[n]W^{0\cdot n} \\ &= 1 + 3 + 2 = 6 \\ X[1] &= \sum_{n=0}^{2} x[n]W^{1\cdot n} \\ &= 1\cdot W^0 + 3\cdot W^1 + 2\cdot W^2 \\ &= 1 + (-1.5 - \frac{3\sqrt{3}}{2}j) + (-1 + \frac{2\sqrt{3}}{2}j) = -1.5 - \frac{\sqrt{3}}{2}j \\ X[2] &= \sum_{n=0}^{2} x[n]W^{2\cdot n} \\ &= 1\cdot W^0 + 3\cdot W^2 + 2\cdot W^4 \\ &= 1 + (-1.5 + \frac{3\sqrt{3}}{2}j) + (-1 - \frac{2\sqrt{3}}{2}j) = -1.5 + \frac{\sqrt{3}}{2}j \end{aligned}$$

Remark. Notice that

- DFT is discrete in frequency domain (DTFT is continuous)
- N-point DFT of a real signal is (very often) complex
- if N-point DFT is real-valued then $x[n]$ has to be “symmetric”
- each value of $X[k]$ is a dot product of $x[n]$ and W with some constant angle jump (nk)
- $X[0]$ is the sum of values of $x[n]$ (DC-component)
- values of $X[k]$ are N-periodic: $X[k] = X[k+N] = X[k+2N] = \dots$
- absolute values (amplitude spectrum) are even $|X[1]| = |X[-1]|$
- angle values are odd $\angle X[1] = -\angle X[-1]$

Discrete Fourier transform is a linear operation. It can be calculated in matrix form as (*Mitra 2Ed Sec. 3.2.2 / 3Ed Sec. 5.2.2*)

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where $\mathbf{X}$ is a column vector of the N frequency-domain DFT samples, $\mathbf{x}$ is a column vector of N time-domain input samples, and $\mathbf{D}_N$ is the $N \times N$ DFT matrix (`dftmtx` in Matlab)

$$\begin{aligned}
\mathbf{X} &= [X[0]\ X[1] \dots\ X[N-1]]^T \\
\mathbf{x} &= [x[0]\ x[1] \dots\ x[N-1]]^T
\end{aligned}$$

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^1 & W_N^2 & \dots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

In this problem `dftmtx` gives as expected

$$\mathbf{D}_3 = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 \\ 1.0000 & -0.5000 - 0.8660i & -0.5000 + 0.8660i \\ 1.0000 & -0.5000 + 0.8660i & -0.5000 - 0.8660i \end{bmatrix}$$

In the inverse DFT $\mathbf{x} = \mathbf{D}_N^{-1}\mathbf{X}$ the matrix $\mathbf{D}_N^{-1}$ is

$$\mathbf{D}_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \dots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \dots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \dots & W_N^{-(N-1)(N-1)} \end{bmatrix}$$

It can be seen that $\mathbf{D}_N^{-1} = (1/N)\mathbf{D}_N^*$.

30. **Problem:** Let $g[n]$ and $h[n]$ be two finite-length sequences $g[n] = \{\underline{5}, 2, 4\}$ and $h[n] = \{\underline{-3}, 4, 0, 2\}$.

a) Determine the linear convolution $y_L[n] = g[n] \circledast h[n]$.

b) Determine the circular convolution $y_C[n] = g_e[n] \textcircled{4}\, h[n]$, where $g_e[n]$ is extended to length of 4 by zero-padding.

c) Determine the circular convolution $y_C[n] = g_e[n] \textcircled{6}\, h_e[n]$, where both sequences are extended to length of 6 by zero-padding.

Solution: In this problem linear convolution $y_L[n]$ (*Mitra 2Ed Sec. 2.5.1 / 3Ed Sec. 2.5.1*) and circular convolution $y_C[n]$ (*Mitra 2Ed Sec. 3.4.2 / 3Ed Sec. 5.4.2*) are computed using sequences $h[n] = \{\underline{5},\ 2,\ 4\}$ and $x[n] = \{\underline{-3},\ 4,\ 0,\ 2\}$.

Linear convolution $y[n] = h[n] \circledast x[n]$ can be computed using “flip and slide” method in Figure 48(a). $x[n]$ is flipped and at each n the items are multiplied and finally all summed together. In the figure, when $n = 1$, it gives $y_L[1] = h[0]x[1-0] + h[1]x[1-1] + h[2]x[1-2] = 20 - 6 + 0 = 14$.

Computation of circular convolution $y_C[n] = h[n] \textcircled{N} x[n]$ can be illustrated with “a circular buffer” of length N in Figure 48(b). $x[n]$ is flipped and at each n the items are multiplied. There are always N terms to be added to get the result at n. In the figure, when $N = 4$ and $n = 1$, it gives $y_C[1] = h[0]x[1] + h[1]x[0] + h[2]x[3] + h[3]x[2] = 20 - 6 + 8 + 0 = 22$.

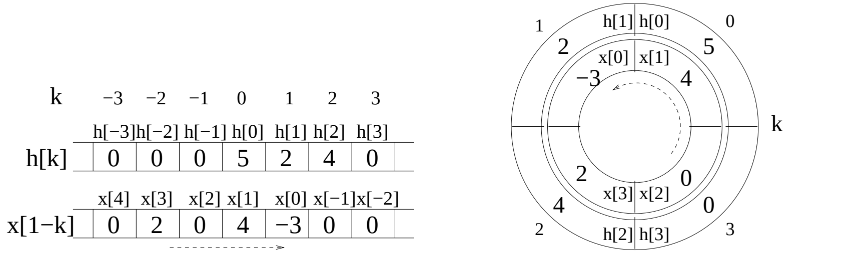


Figure 48: Problem 30: Convolution depicted with “flip and slide” method, (a) left, linear convolution, (b) right, circular convolution with $N = 4$. As an example, in both cases the convolution sum is computed at $n = 1$.

a) Linear convolution: $y_L[n] = \sum_{k=0}^{2} h[k]x[n-k]$. Its length will be $L\{h[n]\} + L\{x[n]\} - 1 = 6$. Using “flip around and slide”:

$$\begin{aligned}
y_L[0] &= h[0]x[0] = 5 \cdot (-3) = -15 \\
y_L[1] &= h[0]x[1] + h[1]x[0] = 5 \cdot 4 + 2 \cdot (-3) = 14 \\
y_L[2] &= h[0]x[2] + h[1]x[1] + h[2]x[0] = 5 \cdot 0 + 2 \cdot 4 + 4 \cdot (-3) = -4 \\
y_L[3] &= h[0]x[3] + h[1]x[2] + h[2]x[1] = 5 \cdot 2 + 2 \cdot 0 + 4 \cdot 4 = 26 \\
y_L[4] &= h[1]x[3] + h[2]x[2] = 2 \cdot 2 + 4 \cdot 0 = 4 \\
y_L[5] &= h[2]x[3] = 4 \cdot 2 = 8
\end{aligned}$$

Therefore,

$$y_L[n] = \{\underline{-15},\ 14,\ -4,\ 26,\ 4,\ 8\}$$

b) Circular convolution is computed in $N = 4$ points

$$y_C[n] = h_e[n] ④ x[n] = \sum_{k=0}^{3} h_e[k]x[< n-k >_4]$$

where $h_e[n] = \{\underline{5},\ 2,\ 4,\ 0\}$ is zero-extended version of $h[n]$, and $< n >_4$ is modulo-4 operation. Hence, $h[< n-5 >_4] = h[< n-1 >_4]$, i.e. the sequence can be thought to be periodic with the period $\{h[0], h[1], h[2], h[3]\}$.

$$\begin{aligned}
y_C[0] &= h_e[0]x[< 0-0 >_4] + h_e[1]x[< 0-1 >_4] + \\
&\quad h_e[2]x[< 0-2 >_4] + h_e[3]x[< 0-3 >_4] \\
&= h_e[0]x[0] + h_e[1]x[3] + h_e[2]x[2] + h_e[3]x[1] \\
&= 5 \cdot (-3) + 2 \cdot 2 + 4 \cdot 0 + 0 \cdot 4 = -11 \\
y_C[1] &= h_e[0]x[1] + h_e[1]x[0] + h_e[2]x[3] + h_e[3]x[2] \\
&= 5 \cdot 4 + 2 \cdot (-3) + 4 \cdot 2 + 0 \cdot 0 = 22 \\
y_C[2] &= h_e[0]x[2] + h_e[1]x[1] + h_e[2]x[0] + h_e[3]x[3] \\
&= 5 \cdot 0 + 2 \cdot 4 + 4 \cdot (-3) + 0 \cdot 2 = -4 \\
y_C[3] &= h_e[0]x[3] + h_e[1]x[2] + h_e[2]x[1] + h_e[3]x[0] \\
&= 5 \cdot 2 + 2 \cdot 0 + 4 \cdot 4 + 0 \cdot (-3) = 26
\end{aligned}$$

Thus,

$$y_C[n] = \{\underline{-11},\ 22,\ -4,\ 26\}$$

c) Circular convolution using $N = 6$ points

$$y_C[n] = h_e[n] ⑥ x_e[n] = \sum_{k=0}^{5} h_e[k]x[< n-k >_6]$$

where $h_e[n] = \{\underline{5},\ 2,\ 4,\ 0,\ 0,\ 0\}$, and $x_e[n] = \{\underline{-3},\ 4,\ 0,\ 2,\ 0,\ 0\}$ are zero-padded versions. Computing like in (b) the result is

$$y_C[n] = \{\underline{-15},\ 14,\ -4,\ 26,\ 4,\ 8\} \equiv y_L[n]$$

If N in circular convolution is chosen so that $N \geq L\{y_L[n]\}$, then $y_C[n] = y_L[n]$.

Remark. Circular convolution has a close connection to Discrete Fourier Transform (DFT). For example, in (b)

$$y_C[n] = h_e[n] ④ x[n] \quad \xrightarrow{\text{DFT-4}} \quad H_e[k] \cdot X[k] = Y_C[k] \quad \xrightarrow{\text{IDFT-4}} \quad y_C[n]$$

31. **Problem:** Consider a LTI system depicted in Figure 49. (a) Difference equation? (b) Compute $X(z)$ when $x[n] = (-0.8)^n\mu[n]$. (c) Transfer function $H(z)$? (d) Compute $y[n]$.

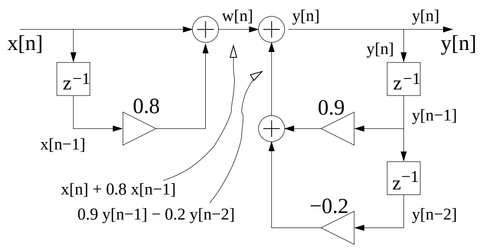


Figure 49: LTI system of Problem 31.

Solution:

a) The input-output-relation is $y[n] - 0.9y[n-1] + 0.2y[n-2] = x[n] + 0.8x[n-1]$. Notice that the coefficients in the diagram are also present in the difference equation (past output values maybe as opposite numbers).

b) If computing using the definition, see Problem 28(b). From the z-transform table directly:

$$\begin{aligned}
Z\{a^n\mu[n]\} &= \frac{1}{1-az^{-1}} \\
(-0.8)^n\mu[n] &\leftrightarrow \frac{1}{1+0.8z^{-1}}
\end{aligned}$$

c) Using the z-transform pair $K \cdot w[n-n_0] \leftrightarrow K \cdot z^{-n_0}W(z)$:

$$\begin{aligned}
y[n] - 0.9y[n-1] + 0.2y[n-2] &= x[n] + 0.8x[n-1] && | \ z\text{-transform} \\
Y(z) - 0.9z^{-1}Y(z) + 0.2z^{-2}Y(z) &= X(z) + 0.8z^{-1}X(z) \\
Y(z)\big(1 - 0.9z^{-1} + 0.2z^{-2}\big) &= X(z)\big(1 + 0.8z^{-1}\big) \\
Y(z) &= X(z)\frac{1+0.8z^{-1}}{1-0.9z^{-1}+0.2z^{-2}} && |/X(z) \\
H(z) = Y(z)/X(z) &= \frac{1+0.8z^{-1}}{1-0.9z^{-1}+0.2z^{-2}}
\end{aligned}$$

The flow (block) diagram was given in direct form (DF) (*Mitra 2Ed Sec. 6.4.1 / 3Ed Sec. 8.4.1*). The coefficients of the diagram are that of the difference equation and transfer function. Coefficients in the loop (IIR subfilter) are in the denominator polynomial and coefficients of the FIR part can be found in the numerator polynomial.

d) Using convolution theorem

$$\begin{aligned}
Y(z) &= H(z) \cdot X(z) \\
&= \frac{1+0.8z^{-1}}{1-0.9z^{-1}+0.2z^{-2}} \cdot \frac{1}{1+0.8z^{-1}} \\
&= \frac{1}{1-0.9z^{-1}+0.2z^{-2}} && | \text{ partial fraction expansion} \\
&= \frac{5}{1-0.5z^{-1}} + \frac{-4}{1-0.4z^{-1}} && | \text{ inverse } z\text{-transform} \\
y[n] &= 5 \cdot (0.5)^n\mu[n] - 4 \cdot (0.4)^n\mu[n]
\end{aligned}$$

32. **Problem:** Consider the pole-zero plots in Figure 50.

   a) What is the order of each transfer function?
   b) Are they FIR or IIR?
   c) Sketch the amplitude response for each filter.
   d) What could be the transfer function of each filter?

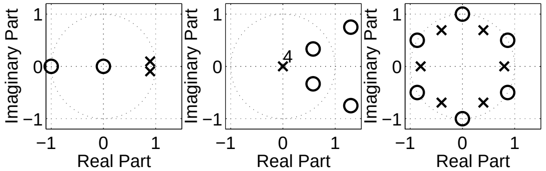


Figure 50: Pole-zero plots of LTI systems in Problem 32.

Solution: The z-transform of the impulse response $h[n]$ of the LTI system is the transfer function $H(z)$ (with certain regions of convergence, ROCs, see Problem 34). It can be written as a rational function in z^{-1} as follows

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= K \cdot \frac{(1 - z_1 z^{-1}) \cdot (1 - z_2 z^{-1}) \cdot \ldots \cdot (1 - z_M z^{-1})}{(1 - p_1 z^{-1}) \cdot (1 - p_2 z^{-1}) \cdot \ldots \cdot (1 - p_N z^{-1})} = K \cdot \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

where b_i are the coefficients of the numerator polynomial $B(z)$, and a_i are the coefficients of the denominator polynomial $A(z)$. The order of $H(z)$ is $\max\{M, N\}$.

Those points z_i where $B(z) = 0$ are called "zeros", and points p_i where $A(z) = 0$ are called "poles". The figure with zeros (circles) and poles (crosses) plotted in the complex plane is called "pole-zero plot" (diagram) of the transfer function.

The rules of thumb for determining amplitude response from the pole-zero-diagram (*Mitra 2Ed Sec. 4.3.4 / 3Ed Sec. 6.7.4*)

1. Examine the frequencies $\omega \in (0 \ldots \pi)$, in other words, the observation point moves on the unit circle counterclockwise from $(1, 0j)$ to $(-1, 0j)$. In each point the amplitude response $|H(e^{j\omega})|$ is estimated. A "simple" function $H(e^{j\omega})$ has a smooth response.
2. The amplification is big, when a pole is close to unit circle (a small factor in denominator) or a zero is far from unit circle. The closer the pole is to unit circle, the narrower the amplification is in frequency area.
3. The amplification is small, when a pole is far from the unit circle (big factors in denominator) or there is a zero close to unit circle.
4. The amplification is zero, if a zero is on the unit circle at observation frequency.
5. Poles or zeros in the origo do not affect at all because the distance is always 1.
6. The amplification cannot be found from pole-zero plot. Normally $H(e^{j\omega})$ is scaled so that the maximum value is set to be 1: $H(e^{j\omega}) \leftarrow H(e^{j\omega}) / \max\{|H(e^{j\omega})|\}$.

a) The order is the maximum of the number of poles or zeros (not in origo).

   So, (i) 2 poles, 1 zero: 2nd order; (ii) 4 zeros: 4th order; (iii) 6 poles, 6 zeros: 6th order.

   Note, in analog $H(s)$ there are only poles, but in digital $H(z)$ there can be both poles and zeros.

b) If there is any pole (cross in the graph) outside the origo, it means that there is at least first-order polynomial in the denominator in $H(z) \Leftrightarrow$ there is a feedback in the system $\Leftrightarrow$ IIR.

   Hence, (i) IIR; (ii) FIR; (iii) IIR.

c) The analysis with graphs is done below for each case separately.

   Shortly, (i) lowpass with narrow passband; (ii) highpass; (iii) a comb filter.

d) The transfer function can be constructed from zeros z_i and poles p_i

$$H(z) = K \cdot \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

   However, the scaling factor K cannot be seen from the pole-zero-plot. Therefore K is set so that $\max\{|H(e^{j\omega})|\} = 1$.

Next, a closer look at (c) is given for each filter.

i) Without computing any exact values of the amplitude response, it is possible to approximate it by looking at the positions of zeros and poles. The angular frequency gets values from 0 to π, and the observation is done on a unit circle counterclockwise.

   Poles are close to unit circle at $\omega = \pm\pi/30$ in Figure 51(a). Therefore the amplitude response gets the maximum approximately at that frequency and the filter is lowpass type, see the sketch in Figure 51(b). The closer the poles are the unit circle, the narrower the maximum area is. The value at $\omega = \pi$ is zero.

   In this case the exact locations of poles and zeros were known ($z_1 = -1$, $p_1 = 0.8950 + 0.0947i$, $p_2 = 0.8950 - 0.0947i$). The actual transfer function is $H(z) = K \cdot (1 + z^{-1})/(1 - 1.79z^{-1} + 0.81z^{-2})$ from which the actual frequency response is received by $z \leftarrow e^{j\omega}$. Some values in range $0 \ldots \pi$ are computed below, and K is chosen so that the maximum of $|H(z)|$ is one. Figures are plotted using Matlab in both linear scale and in logarithmic scale in Figure 51(c) and (d), respectively.

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$	ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
0	1	1	$3\pi/4$	$-.0008 + .0023j$	.0025
$\pi/4$	$-.0277 + .0210j$	.0348	π	0	0
$\pi/2$	$-.0049 + .0061j$	.0078			

ii) There are four zeros in Figure 52(left). At $\omega \approx \pi/6$ the zeros are closest to the observation point, and the minimum of the response is probably reached (bandstop). At $\omega = \pi$ the zeros are much further away than at $\omega = 0$, so the attenuation is much stronger at low frequencies (highpass). Notice that $|H(e^{j0})| \neq 0$, because there is not a zero on the unit circle at $\omega = 0$. The filter can be a highpass or bandstop FIR filter.

   Actually, $H(z) = 1 - 3.753z^{-1} + 5.694z^{-2} - 3.753z^{-3} + z^{-4}$. Filter coefficients have a certain symmetry as well as the zeros lie in a certain symmetry, which implies a linear-phase filter, see Problem 35. The minimum of $|H(e^{j\omega})| \approx 0.0114$ (scaled) at $\omega \approx 0.11\pi$, which is different from $\pi/6$ estimated earlier. All "zero vectors" affect to the response, see the remark text below.

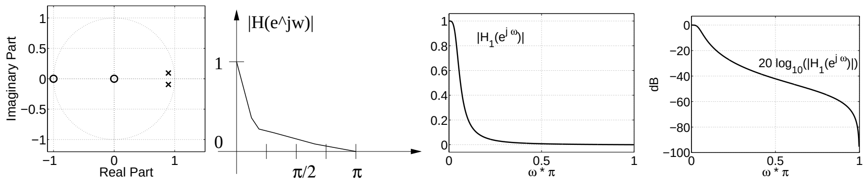


Figure 51: Problem 32(i): (a) Pole-zero-diagram, (b) an example of approximated amplitude response, (c) actual amplitude response $|H_1(e^{j\omega})|$ in linear scale, (d) actual amplitude response $|H_1(e^{j\omega})|$ in decibels.

iii) Zeros are on the unit circle at uniform invervals forcing the amplification drop down to zero, see Figure 52(right). This type of periodic filter is often called a comb filter. The maximum is scaled to one. Note that all poles and zeros affect, so that if there were not exactly same intervals between poles and zeros, the amplitude response would also turn out to be non-symmetric.

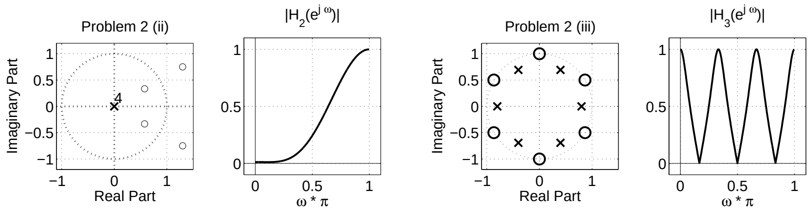


Figure 52: Problem 32(ii),(iii): Pole-zero-diagram and corresponding amplitude response of $|H_2(e^{j\omega})|$ left, and $|H_3(e^{j\omega})|$ right.

Remark. Determining amplitude response from the pole-zero-diagram, theory in background.

Any transfer function $H(z)$ can be expressed in form of

$$H(z) = \frac{|p_0|}{|d_0|} \frac{(1 - z_1 z^{-1})(1 - z_2 z^{-1})...(1 - z_M z^{-1})}{(1 - p_1 z^{-1})(1 - p_2 z^{-1})...(1 - p_N z^{-1})}$$

In order to achieve this, all zeros (z_i) and poles (p_i) of $H(z)$ has to be computed. Zeros are the roots of the numerator polynomial and poles are the roots of the denominator polynomial. Numerator part is "FIR part" (always stable, $y[n]$ depends only on values of $x[n-k_i]$), denominator is "IIR part" (feedback, in order to compute $y[n]$ some old values of it has to be used).

Frequency response is the transfer function computed on unit circle, i.e. substitution $z = e^{j\omega}$:

$$H(e^{j\omega}) = \frac{|p_0|}{|d_0|} \frac{(1 - z_1 e^{-j\omega})(1 - z_2 e^{-j\omega})...(1 - z_M e^{-j\omega})}{(1 - p_1 e^{-j\omega})(1 - p_2 e^{-j\omega})...(1 - p_N e^{-j\omega})}$$

We are interested in amplitude response $|H(e^{j\omega})|$. Because the expression is in a product form, the absolute value of $|H(e^{j\omega})|$ can be computed with its first order blocks. Let $K = |p_0|/|d_0|$, B_i be the length of a first order block in numerator polynomial, and A_i the

length of a first order block in denominator polynomial:

$$|H(e^{j\omega})| = K \cdot \frac{\overbrace{|(1 - z_1 e^{-j\omega})|}^{B_1}\ \overbrace{|(1 - z_2 e^{-j\omega})|}^{B_2} \ldots \overbrace{|(1 - z_M e^{-j\omega})|}^{B_M}}{\underbrace{|(1 - p_1 e^{-j\omega})|}_{A_1}\ \underbrace{|(1 - p_2 e^{-j\omega})|}_{A_2} \ldots \underbrace{|(1 - p_N e^{-j\omega})|}_{A_N}} = K \cdot \frac{\prod_{k=1}^{M} B_k}{\prod_{k=1}^{N} A_k}$$

The frequency axis lies on the unit circle from $\omega = 0$, which is a complex point $e^{j\omega}|_{\omega=0} = 1$ to $\omega = \pi$, which is situated at $e^{j\omega}|_{\omega=\pi} = -1$. The observation frequency ω_0 gets values $0 \ldots \pi$.

B_i is called a "zero vector", i.e. it is the length from the observation point ω_0 to zero i. A_i is a "pole vector" correspondingly.

Any small A_i (pole close to unit circle) gives big value of $|H(e^{j\omega})|$. Any small B_i (zero close to unit circle) decreases $|H(e^{j\omega})|$. However, it should be noticed that $|H(e^{j\omega})|$ is a product of **all** zero vectors and **all** pole vectors.

For example, in Figure 53(a) $M = 2$ and $N = 2$:

$$|H(e^{j\omega})| = K \cdot \frac{\overbrace{|(1 - z_1 e^{-j\omega})|}^{B_1}\ \overbrace{|(1 - z_2 e^{-j\omega})|}^{B_2}}{\underbrace{|(1 - p_1 e^{-j\omega})|}_{A_1}\ \underbrace{|(1 - p_2 e^{-j\omega})|}_{A_2}}$$

It can be roughly estimated that the filter is highpass, because around $\omega = 5\pi/6$ A_1 is smallest and therefore $|H(e^{j\omega})|$ is at maximum. Actually the maximum might be at $\omega = \pi$, where $A_1 \cdot A_2$ is probably smaller. The rough estimate of the amplitude response $(0 \ldots \omega_0 \ldots \pi)$ is given in Figure 53(b).

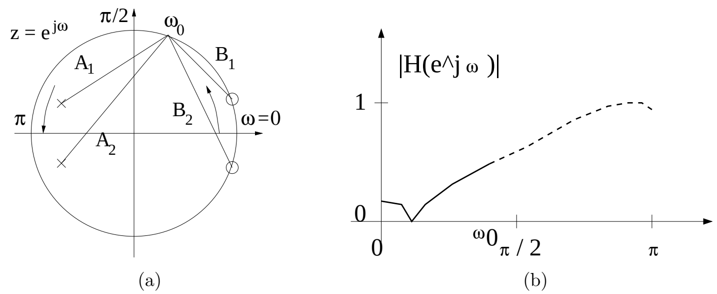


Figure 53: (a) Zero vectors B_k and pole vectors A_k. ω runs $0..\pi$. (b) Amplitude response roughly from the pole-zero-diagram.

The rules of thumb were given on page 67.

It can also be seen that the frequency response in discrete-time domain is always 2π-periodic. Because $|H(e^{j\omega})|$ is an even function, it is only necessary to draw angles $0 \ldots \pi$.

33. **Problem:** Consider the filter described in Figure 54.

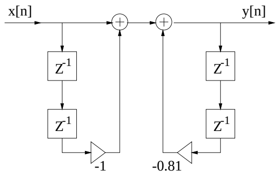


Figure 54: LTI system of Problem 33, also at page 12.

a) Derive the difference equation of the system.

b) Calculate the transfer function $H(z)$.

c) Calculate the zeros and poles of $H(z)$. Sketch the pole-zero plot. Is the system stable and/or causal?

d) Derive frequency response $H(e^{j\omega})$.

e) Sketch the magnitude (amplitude) response $|H(e^{j\omega})|$ roughly. Which frequency gives the maximum value of $|H(e^{j\omega})|$?

f) Compute the equation for the impulse response $h[n]$ using partial fraction expansion and inverse z-transform.

Solution: Notice that the same filter can be represented (i) as a block diagram, (ii) with a difference equation, (iii) with a transfer function (and ROC), (iv) with an impulse response, (v) with poles, zeros and gain.

a) Difference equation: $y[n] = x[n] - x[n-2] - 0.81y[n-2]$

b) Transfer function $H(z)$ can be obtained from $h[n]$ using z-transform pairs:

$$\begin{aligned} Z\{x[n]\} &= X(z) \\ Z\{a \cdot x[n-n_0]\} &= a \cdot z^{-n_0} \cdot X(z). \end{aligned}$$

Hence,

$$\begin{aligned} y[n] &= x[n] - x[n-2] - 0.81y[n-2] \\ Y(z) &= X(z) - z^{-2}X(z) - 0.81z^{-2}Y(z) \\ (1+0.81z^{-2}) \cdot Y(z) &= (1-z^{-2}) \cdot X(z) \\ Y(z) &= X(z) \cdot \frac{1-z^{-2}}{1+0.81z^{-2}} \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{1-z^{-2}}{1+0.81z^{-2}} \end{aligned}$$

c) Zeros are the points, where the numerator of transfer function $H(z)$ is zero:
$1 - z^{-2} = 0 \Leftrightarrow z^2 = 1 \Leftrightarrow z = \pm 1$.
Poles are the points, where the denominator of transfer function $H(z)$ is zero:
$1 + 0.81z^{-2} = 0 \quad \Leftrightarrow \quad z^2 = -0.81 \quad \Leftrightarrow \quad z = \pm 0.9j$
The pole-zero plot of the system is (a common notation is to use a ◯ for a zero and a × for a pole) in Figure 55(a).

The system is **causal**, because current output does not depend on future values of $x[n]$ and $y[n]$ (time-domain view). The system is **stable**, because the impulse response $h[n]$ is absolutely summable (time-domain view).

On the other hand, if all poles in the pole-zero plot are inside the unit circle, i.e., the region of convergence (ROC) includes both the unit circle and the infinity, the filter is causal and stable (see Problem 34).

d) Frequency response of the system $H(e^{j\omega})$ (often for continuous systems $H(\omega)$) is obtained by applying $z = e^{j\omega}$ (continuous $s = j\omega$). If the unit circle is contained in the ROC, it is possible to apply $H(e^{j\omega}) = H(z)|_{z=e^{j\omega}}$:

$$H(e^{j\omega}) = \frac{1-e^{-2j\omega}}{1+0.81e^{-2j\omega}}$$

e) The amplitude response can be computed as exact as wanted using the mathematical functions. It can be computed also in specific points using calculator or computer. These will be explained after the roughest way, which is graphical approximation from poles and zeros.

The sketch the magnitude (amplitude) response $|H(e^{j\omega})|$ can be drawn by using pole-zero plot. There are zeros at $z = 1$ and $z = -1$. The corresponding angular frequencies are 0 and π, because $e^{j0} = 1 + 0j$ and $e^{j\pi} = -1 + 0j$. Hence, amplitude response is zero when $\omega = 0$ and $\omega = \pi$. It is also clear that the maximum value is at $\omega = \pi/2$, where the pole is closest to the unit circle. A sketch is given in Figure 55(b).

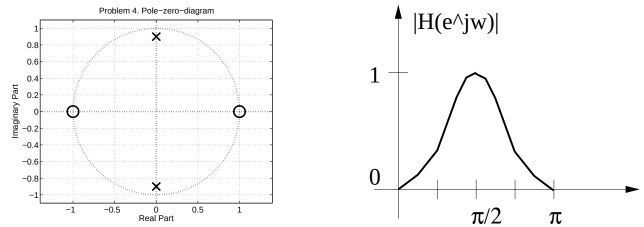


Figure 55: Problem 33: (a) Pole-zero plot of $H(z)$, (b) A rough sketch of amplitude response using pole-zero.

Second, the amplitude response $H(e^{j\omega}) = (1 - e^{-2j\omega})/(1 + 0.81e^{-2j\omega})$ can be calculated in certain poins. More points, more exact amplitude response. Start with points $\omega = \{0, \pi/4, \pi/2, 3\pi/4, \pi\}$, and calculate more if it seems to be appropriate. If your calculator does not support complex exponentials, decompose them by Euler's formula. (Notice that in Matlab you can use directly function `exp`.) A new sketch is drawn in Figure 56.

ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$	ω	$H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$
0	0	0	$5\pi/8$	$0.6352 + 2.5067$	2.5859
$\pi/8$	$0.0199 - 0.4568j$	0.4573	$3\pi/4$	$0.1147 + 1.0929j$	1.0989
$\pi/4$	$0.1147 - 1.0929j$	1.0989	$7\pi/8$	$0.0199 + 0.4568j$	0.4573
$3\pi/8$	$0.6352 - 2.5067j$	2.5859	π	0	0
$\pi/2$	$10.5263 - 0.0000j$	10.5263			

Third, the magnitude response can (only sometimes) be simplified. For example, this time the simplified version is relatively simple. Simplification is sometimes needed

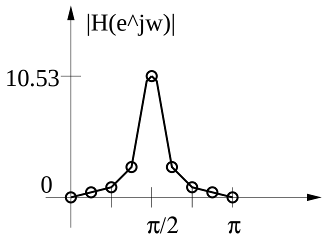


Figure 56: A sketch of amplitude response after computing several values in Problem 33(e).

to some proofs, etc.

$$\begin{aligned} |H(e^{j\omega})|^2 &= H(e^{j\omega})H^\star(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega}) && \mid \text{complex conjugate} \\ &= \frac{1-e^{-2j\omega}}{1+0.81e^{-2j\omega}} \cdot \frac{1-e^{+2j\omega}}{1+0.81e^{+2j\omega}} \\ &= \frac{1+1-(e^{2j\omega}+e^{-2j\omega})}{1+0.81^2+0.81(e^{2j\omega}+e^{-2j\omega})} \\ &= \frac{2-2\cos(2\omega)}{1.6561+1.62\cos(2\omega)} && \mid \text{square} \end{aligned}$$

$|H(e^{j\omega})|$ gets the maximum value at frequency $\omega = \frac{\pi}{2}$. The maximum value is

$$|H(e^{j\omega})|_{max} = \left|H(e^{j\frac{\pi}{2}})\right| \approx 10.53$$

f) Notice that the partial fraction expansion can be written in various forms, see Problem 9, for instance. The transform pair $a^n\mu[n] \leftrightarrow \frac{1}{1-az^{-1}}$ is applied again.

$$\begin{aligned} H(z) &= \frac{1-z^{-2}}{1+0.81z^{-2}} \\ &= \frac{1}{1+0.81z^{-2}} - z^{-2}\cdot\frac{1}{1+0.81z^{-2}} && \mid \text{part. frac. exp.} \\ &= [\frac{0.5}{1-0.9jz^{-1}} + \frac{0.5}{1+0.9jz^{-1}}] - z^{-2}[\frac{0.5}{1-0.9jz^{-1}} + \frac{0.5}{1+0.9jz^{-1}}] \end{aligned}$$

$$\begin{aligned} h[n] &= 0.5\cdot((0.9j)^n\mu[n] + (-0.9j)^n\mu[n]) - \\ &\quad 0.5\cdot\left((0.9j)^{n-2}\mu[n-2] + (-0.9j)^{n-2}\mu[n-2]\right) \\ &\approx \{\underline{1.0000},\ 0,\ -1.8100,\ 0,\ 1.4661,\ 0,\ -1.1875,\ \ldots\} \end{aligned}$$

Matlab (`residuez`) may give a different form of the same sequence:

$$h[n] \approx -1.2346\cdot\delta[n] + 1.1173\cdot(0.9j)^n\mu[n] + 1.1173\cdot(-0.9j)^n\mu[n]$$

34. **Problem:** The transfer function of a filter is

$$H(z) = \frac{1-z^{-1}}{1-2z^{-1}+0.75z^{-2}}$$

a) Compute the zeros and poles of $H(z)$.
b) What are the three different regions of convergence (ROC)?
c) Determine the ROC and the impulse response $h[n]$ so that the filter is causal.
d) Determine the ROC and the impulse response $h[n]$ so that the filter is stable.

Solution: Let us begin by reviewing some properties (*Mitra 2Ed Sec. 3.8 / 3Ed Sec. 6.3*)

- The filter is causal $\Leftrightarrow \infty$ belongs to the region of convergence (ROC).
- The filter is stable $\Leftrightarrow$ unit circle belongs to ROC, $H(z)$ converges on the unit circle.
- ROC on z-plane must not contain any poles; it may be a ring between two poles, the disc limited by the closest pole to origin or the plane outside the most distant pole from origin.
- It is easiest to do the the inverse z-transform (here) by calculating first the fractional expansion of the $H(z)$ and then inverting each part of it individually using the sum of a geometric series.
- The sum of a geometric series is

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}\ , \quad |q| < 1$$

- The z-transform of $h[n]$ is

$$\sum_{n=-\infty}^{\infty} h[n]z^{-k}$$

a) First we have to solve the poles and zeros:

$$H(z) = \frac{1-z^{-1}}{1-2z^{-1}+0.75z^{-2}} = \frac{z(z-1)}{z^2-2z+0.75}$$

Poles:

$$z^2-2z+0.75 = 0 \Leftrightarrow z = \frac{2\pm\sqrt{4-4\cdot 0.75}}{2} \Leftrightarrow z_1 = 0.5, z_2 = 1.5$$

Zeros:

$$z(z-1) = 0 \Leftrightarrow z_1 = 0, z_2 = 1$$

b) Now we may answer to the questions about stability and causality using different ROCs, see Figure 57:

i) If we require causality, the region of convergence has to include $z=\infty$ Thus, the region of convergence has to be "outside" the pole $z = 1.5$, that is $|z| > 1.5$.
ii) If we require stability, the unit circle has to be on the region of convergence. Thus the region is a ring between the poles: $0.5 < |z| < 1.5$
iii) If ROC is the inner cicle $|z| < 0.5$, we will have a noncausal and astable filter.

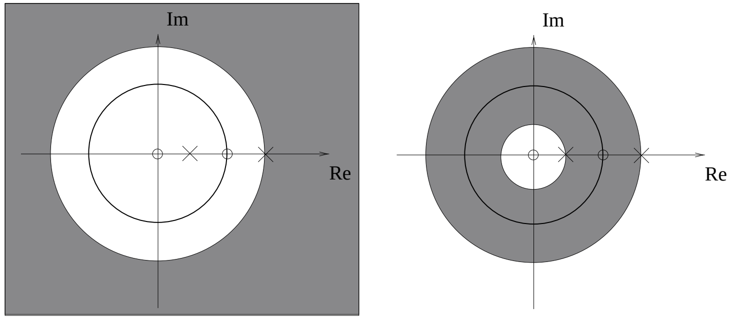


Figure 57: Region of convergence (ROC) in gray in Problem 34: (i) ∞ belongs to ROC - causal filter, (ii) unit circle belongs to ROC - stable filter.

Note, that in this case we cannot have a filter that is both causal and stable.

At this point, when we calculate the impulse response $h[n]$, we have to do an inverse z-transformation for the transfer function $H(z)$. To do this we express the $H(z)$ as a partial fraction expansion as then we may apply the formula of the sum of a geometric series.

Using the poles and zeros we may write the transfer function as follows:

$$H(z) = \frac{(1-z_1z^{-1})(1-z_2z^{-1})}{(1-p_1z^{-1})(1-p_2z^{-1})} = \frac{1-z^{-1}}{(1-0.5z^{-1})(1-1.5z^{-1})}$$

$$\Leftrightarrow \quad H(z) = \frac{A}{1-0.5z^{-1}} + \frac{B}{1-1.5z^{-1}}$$
$$\Leftrightarrow \quad 1 - z^{-1} \equiv A(1-1.5z^{-1}) + B(1-0.5z^{-1})$$

We solve A and B by letting $z \to 0.5$ and $z \to 1.5$

$$z \to 0.5: \quad 1 - 0.5^{-1} = A(1-1.5\cdot 0.5^{-1}) + B(\underbrace{1-0.5\cdot 0.5^{-1}}_{=0})$$
$$\Rightarrow \qquad A = 0.5$$
$$z \to 1.5: \quad 1 - 1.5^{-1} = A(\underbrace{1-1.5\cdot 1.5^{-1}}_{=0}) + B(1-0.5\cdot 1.5^{-1})$$
$$\Rightarrow \qquad B = 0.5$$

Now we may write the expansion

$$H(z) = \frac{0.5}{1-0.5z^{-1}} + \frac{0.5}{1-1.5z^{-1}}$$

c) Causal filter $\Rightarrow$ we know that $|z| > 1.5$. We notice that both fractions in

$$H(z) = \frac{0.5}{1-0.5z^{-1}} + \frac{0.5}{1-1.5z^{-1}}$$

represent a sum of a geometric series, as $|0.5z^{-1}| < 1$ and $|1.5z^{-1}| < 1$ as required. We conclude

$$h_{causal}[n] = Z^{-1}\{H(z)\} = 0.5\cdot 0.5^n\mu[n] + 0.5\cdot 1.5^n\mu[n]$$

See Figure 58(a), the impulse response grows to infinity, i.e. it is not absolutely summable, and therefore the filter is not stable with the criterion $\sum_n |h[n]| < \infty$.

d) Stable filter $\Rightarrow$ we know that $0.5 < |z| < 1.5$. We note that $\sum_{n=0}^{\infty} 1.5^n z^{-n}$ does not converge as $|\frac{1.5}{z}| \geq 1$. We have to convert the expression to higher terms in order to get the denominator to suitable form:

$$\begin{aligned}
H_{p2}(z) &= \frac{1}{2}\cdot\frac{1}{1-(3/2)z^{-1}} \qquad \Big| \quad \cdot(-(2/3)z)/(-(2/3)z) \\
&= -\frac{1}{3}z\frac{1}{1-(2/3)z} \\
&= -\frac{1}{3}z\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n z^n \\
&= -\frac{1}{3}\sum_{n=-\infty}^{\infty}\left(\frac{2}{3}\right)^n \mu[n]z^{n+1} \quad | \text{ let } -m = n+1 \\
&= -\frac{1}{3}\sum_{m=-\infty}^{\infty}\left(\frac{2}{3}\right)^{-m-1} \mu[-m-1]z^{-m}
\end{aligned}$$

Thus, the inverse transform of $H(z)$ is

$$h_{stable}[n] = 0.5\cdot 0.5^n\mu[n] - \frac{1}{3}\left(\frac{2}{3}\right)^{-n-1}\mu[-n-1]$$

which is plotted in Figure 58(b). The impulse response is non-zero for indices $n < 0$, and the filter is not causal with criterion $h[n] < 0,\ n < 0$. The filter is stable.

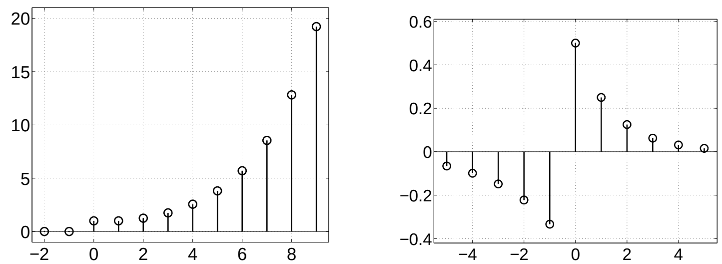


Figure 58: Problem 34: Left, 34(c) where ROC: $|z| > 1.5 \Leftrightarrow$ filter is causal but not stable. Right, 34(d) where ROC: $0.5 < |z| < 1.5 \Leftrightarrow$ filter is not causal but stable.

Remark. In practice, we operate with causal and stable filters, which means that all poles should be inside the unit circle.

35. **Problem:** Examine the following five filters and connect them at least to one of the following categories (a) zero-phase, (b) linear-phase, (c) allpass, (d) minimum-phase, (e) maximum-phase.

$$
\begin{aligned}
h_1[n] &= -\delta[n+1] + 2\delta[n] - \delta[n-1] \\
H_2(z) &= \frac{1 + 3z^{-1} + 2.5z^{-2}}{1 - 0.5z^{-1}} \\
y_3[n] &= 0.5y_3[n-1] + x[n] + 1.2x[n-1] + 0.4x[n-2] \\
H_4(z) &= \frac{0.2 - 0.5z^{-1} + z^{-2}}{1 - 0.5z^{-1} + 0.2z^{-2}} \\
H_5(e^{j\omega}) &= -1 + 2e^{-j\omega} - e^{-2j\omega}
\end{aligned}
$$

Solution: Types of transfer functions are explained in (*Mitra 2Ed Sec. 4.4, 4.6, 4.7, 4.8 / 3Ed Sec. 7.1, 7.2, 7.3*). After some work at least the following pairs can be mentioned: (a) $h_1[n]$, (b) $H_5(e^{j\omega})$, (c) $H_4(e^{j\omega})$, (d) $y_3[n]$, and (e) $H_2(z)$.

If the coefficients of the transfer function are real-valued (as they are in this course), then the pole and zero pairs must be complex conjugates: $z_1 = re^{j\theta}$, $z_2 = re^{-j\theta}$.

If the coefficients of the FIR filter are symmetric, Type I, II, III, and IV, (*Mitra 2Ed Sec. 4.4.3, 4.4.4 / 3Ed Sec. 7.3*) and (*Mitra 2Ed Fig. 4.14, 4.16 / 3Ed Fig. -, 7.17*), then the filter has linear phase response (or even zero-phase). The group delay ($\tau(\omega) = -d/d\omega\ \angle H(e^{j\omega})$) of linear-phase filters is constant for all frequencies.

Another important term is mirror-symmetry respect to the unit circle. In this case the connection between poles or zeros is: $z_1 = re^{j\theta}$, $z_2 = (1/r)e^{j\theta}$ (and their complex conjugates).

For each filter type there is also another example. There are four figures a row for each example, (i) impulse response, (ii) pole-zero-diagram, (iii) amplitude response in desibels and x-axis in range $0 \ldots \pi$, (iv) phase response.

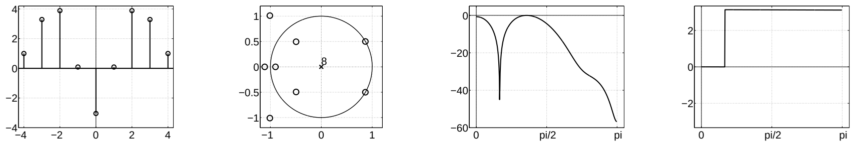
h1) This noncausal FIR filter has zero phase. The impulse response $h_1 = -\delta[n+1] + 2\delta[n] - \delta[n-1]$ is symmetric around the origo in the time-domain. The frequency response can be written

$$
\begin{aligned}
H_1(e^{j\omega}) &= -e^{j\omega} + 2 - e^{-j\omega} = 2 - 2\cos(\omega) \\
|H_1(e^{j\omega})| &= |2 - 2\cos(\omega)| \geq 0 \qquad | \text{ ampl.resp.} \in \mathbf{R} \\
\angle H_1(e^{j\omega}) &= 0 \qquad | \text{ phase resp.} \\
-\frac{d}{d\omega}\angle H_1(e^{j\omega}) &= 0 \qquad | \text{ no delay at all}
\end{aligned}
$$

from which it can be seen that $H_1(e^{j\omega})$ is real-valued. The phase response and group delay ($\tau(\omega) = -d/d\omega\ \angle H(e^{j\omega})$) is therefore zero (or 180 degrees for negative values of $H(e^{j\omega})$) for all frequencies, in other words, the filter is zero-phase (*Mitra 2Ed Sec. 4.4.2 / 3Ed Sec. 7.2.1*) and the signal is not delayed in the filter. Matlab command `filtfilt` can be applied instead of `filter`.

Another example, see Figure 59. The zeros are situated mirror-symmetrically according to the unit circle, and the impulse response and the transfer function are

$$
\begin{aligned}
h[n] &= \{1, 3.2893, 3.8875, 0.0884, \underline{-3.0407}, 0.0884, 3.8875, 3.2893, 1\} \\
H(z) &= \sum_n h[n]z^{-n}
\end{aligned}
$$



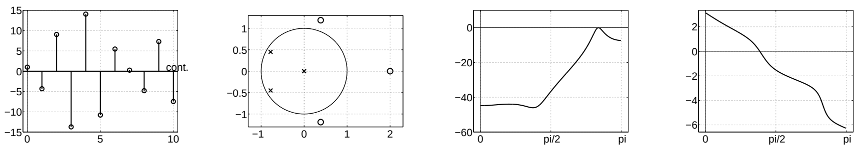
(i) (ii) (iii) (iv)

Figure 59: An example of a zero-phase transfer function in Problem 35. Subfigures (for Figures 59..64), (i) impulse response $h[n]$, (ii) pole-zero plot, (iii) amplitude response $|H(e^{j\omega})|$, x-axis $(0 \ldots \pi)$, (iv) phase response $\angle H(e^{j\omega})$, x-axis $(0 \ldots \pi)$.

H2) When all zeros are outside the unit circle, the filter has maximum phase. The filter is IIR, the two zeros are outside the unit circle. When plotting the amplitude response, it can be noticed that the filter is lowpass (LP). The filter $H_2(z)$ is at least maximum-phase.

Another example on a maximum-phase transfer function (*Mitra 2Ed Sec. 4.7 / 3Ed Sec. 7.2.3*), whose all zeros lie outside the unit circle in Figure 60

$$H(z) = \frac{1 - 2.773z^{-1} + 3.108z^{-2} - 3.125z^{-3}}{1 + 1.559z^{-1} + 0.81z^{-2}}$$



(i) (ii) (iii) (iv)

Figure 60: An example of a maximum-phase transfer function in Problem 35.

y3) When all zeros are inside the unit circle, the filter has minimum phase. From the difference equation we get

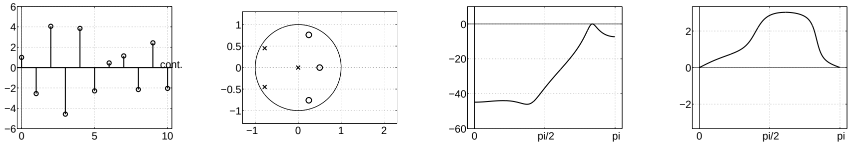
$$H_3(z) = \frac{1 + 1.2z^{-1} + 0.4z^{-2}}{1 - 0.5z^{-1}}$$

The transfer function is similar to $H_2(z)$, but the zeros are now mirror-symmetric to those. Therefore the amplitude response is the same, but the filter is minimum-phase (*Mitra 2Ed Sec. 4.7 / 3Ed Sec. 7.2.3*).

Another example on a minimum-phase transfer function whose all zeros lie inside the unit circle in Figure 61

$$H(z) = \frac{1 - 0.9944z^{-1} + 0.8872z^{-2} - 0.32z^{-3}}{1 + 1.559z^{-1} + 0.81z^{-2}}$$

A minimum-phase transfer function can be converted to a maximum-phase transfer function (or vice versa) by mirroring the zeros respect to the unit circle. This can be done using an appropriate allpass function.



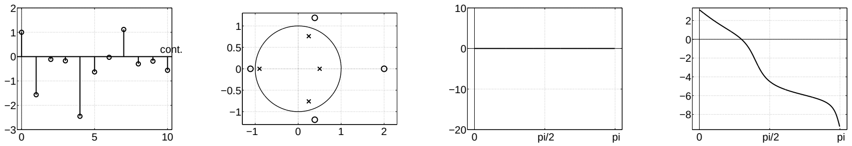
(i) (ii) (iii) (iv)

Figure 61: An example of a minimum-phase transfer function in Problem 35.

H4) If the amplitude response ($z \leftarrow e^{j\omega}$) is $|H(e^{j\omega})| = 1$ for all frequencies, then the filter is allpass (*Mitra 2Ed Sec. 4.6 / 3Ed Sec. 7.1.3*). The phase response differs from filter to filter. Allpass-filters contain both zeros and poles mirror-symmetrically, and there is a certain symmetry in the coefficients of numerator and denominator polynomials, too. Note that gain cannot be seen from the pole-zero plot.

In Figure 62 an allpass transfer function

$$H(z) = -3.4722 \cdot \frac{-0.288 + 0.4785z^{-1} - 0.007771z^{-2} - 0.09443z^{-3} + z^{-4}}{1 - 0.09443z^{-1} - 0.007771z^{-2} + 0.4785z^{-3} - 0.288z^{-4}}$$



(i) (ii) (iii) (iv)

Figure 62: An example of an allpass transfer function in Problem 35.

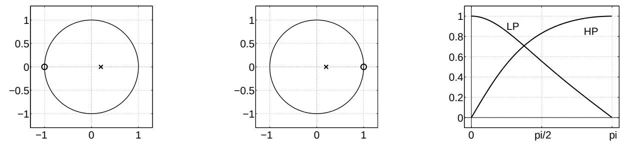
A complemantary transfer function (*Mitra 2Ed Sec. 4.8 / 3Ed Sec. 7.5*) can be obtained using allpass filters. An example of a lowpass filter

$$\begin{aligned} H_{LP}(z) &= 0.5\big(A_0(z) + A_1(z)\big) \\ &= 0.5\Big(1 + \frac{-a + z^{-1}}{1 - az^{-1}}\Big) \\ &= 0.5\Big(\frac{1 - a + z^{-1} - az^{-1}}{1 - az^{-1}}\Big) \end{aligned}$$

where $A_0(z)$ and $A_1(z)$ are allpass transfer functions and its power-complementary highpass filter

$$\begin{aligned} H_{HP}(z) &= 0.5\big(A_0(z) - A_1(z)\big) \\ &= 0.5\Big(1 - \frac{-a + z^{-1}}{1 - az^{-1}}\Big) \\ &= \frac{1+a}{2} \cdot \frac{1 - z^{-1}}{1 - az^{-1}} \end{aligned}$$

In Figure 63(iii) is shown that $|H_{LP}(z)|^2 + |H_{HP}(z)|^2 = 1$, as expected by the definition of power-complementary transfer functions.



LP HP (iii)

Figure 63: An example of power-complementary LP and HP filters in Problem 35.

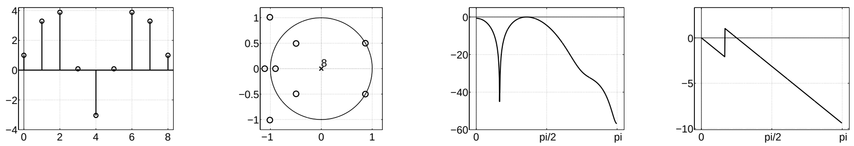
H5) Linear-phase. This impulse response is a shifted (delayed) version of $h_1[n]$. The frequency response is not any more real-valued, but still the phase response is linear and the group delay constant.

$$\begin{aligned} H_5(e^{j\omega}) &= e^{-j\omega} \cdot H_1(e^{j\omega}) \\ |H_5(e^{j\omega})| &= |H_1(e^{j\omega})| = |2 - 2\cos(\omega)| \\ \angle H_5(e^{j\omega}) &= -\omega \qquad | \quad \text{linear} \\ -\frac{d}{d\omega}\angle H_5(e^{j\omega}) &= 1 \qquad | \quad \text{constant} \end{aligned}$$

There are four types of linear-phase transfer functions (*Mitra 2Ed Sec. 4.4.3 / 3Ed Sec. 7.3*). Impulse response of Type 1 is symmetric and odd-length. Type 2 is symmetric and even-length. Type 3 is antisymmetric and odd-length. Type 4 is antisymmetric and even-length. The zeros have mirror-image symmetry respect to the unit circle.

In Figure 64 there is a Type 1 (length: 9, order: 8) impulse response, which is a shifted version of the filter in Figure 59.

$$\begin{aligned} h[n] &= \{\underline{1}, 3.2893, 3.8875, 0.0884, -3.0407, 0.0884, 3.8875, 3.2893, 1\} \\ H(z) &= \sum_n h[n]z^{-n} \end{aligned}$$



(i) (ii) (iii) (iv)

Figure 64: An example of a linear-phase transfer function in Problem 35.

36. **Problem:** Show that the periodic impulse train $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$ can be expressed as a Fourier series $p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{j\Omega_T kt}$, where $\Omega_T = 2\pi/T$ is the sampling angular frequency.

Solution: Since $p(t)$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

is a periodic function of time t with a period T (time between samples), it can be represented as Fourier series (F-series for periodic, F-transform for non-periodic signals):

$$p(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{j(2\pi nt/T)}$$

where Fourier coefficients (note, $p(t)$ over one period T)

$$c_n = \frac{1}{T}\int_T p(t)\, e^{-j(2\pi nt/T)}\,\mathrm{d}t$$

The unit impulse function (continous-time) has properties
(1) $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$, and
(2) $\int_{-\infty}^{\infty} \delta(t)a(t)\,\mathrm{d}t = a(t)|_{t=0}$.
Therefore Fourier series coefficients are:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-j(2\pi nt/T)} dt = \frac{1}{T} e^{-j(2\pi nt/T)}|_{t=0} = \frac{1}{T}$$

Hence

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{j(2\pi nt/T)}$$

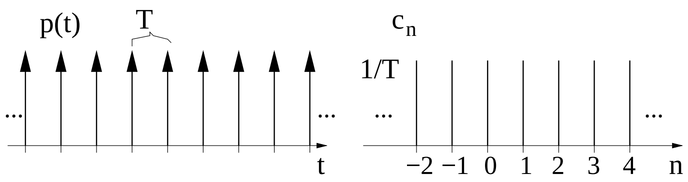


Figure 65: Problem 36: impulse train $p(t)$ left, and its Fourier series coefficients c_n right.

37. **Problem:** Impulse train in Problem 36 can be also expressed as a Fourier transform $P(j\Omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\,\Omega_s)$. Sampling can be modelled as multiplication in time domain $x[n] = x_p(t) = x(t)p(t)$. What is $X_p(j\Omega)$ for an arbitrary input spectrum $X(j\Omega)$?

Solution: The Fourier series of a continuous-time signal can be expressed

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \, e^{jk\Omega_0 t}$$

where a_k are Fourier coefficients and Ω_0 is fundamental angular frequency. Fourier transform of a periodic signal can be written in form of

$$X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\Omega - k\Omega_0)$$

So, the impulse train $p(t)$ of Problem 36 with all coefficients $a_k = 1/T_s$ and fundamental angular frequency Ω_s can be written as

$$P(j\Omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

Sampling in time and frequency domain can be modeled $x[n] = x(t)\cdot p(t) \leftrightarrow \frac{1}{2\pi}\big[X(j\Omega) \circledast P(j\Omega)\big]$, which finally gives

$$\begin{aligned}
\frac{1}{2\pi}\big[P(j\Omega) \circledast X(j\Omega)\big] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\theta)X(j(\Omega-\theta))\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\theta - k\Omega_s)X(j(\Omega-\theta))\mathrm{d}\theta \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(\theta - k\Omega_s)X(j(\Omega-\theta))\mathrm{d}\theta \quad \Big| \quad \int \delta(t)x(t)dt = x(t)|_{t=0} \\
&= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))
\end{aligned}$$

In other words, the spectrum $X(e^{j\omega})$ of the discrete-time signal $x[n]$ can be obtained by summing the shifted spectra $X(j\Omega)$ of the corresponding analog signal $x(t)$. Spectra $X(j\Omega)$ are scaled by $(1/T_s)$ and copied at every sampling (angular) frequency.

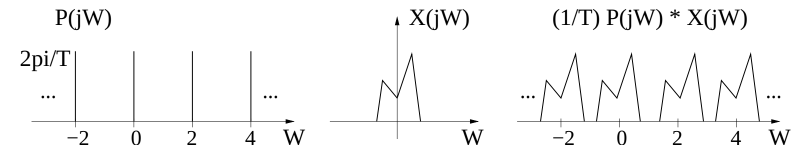


Figure 66: Problem 37: Left, an example of a spectrum $P(j\Omega)$ of an impulse train, middle, a spectrum $X(j\Omega)$ of an arbitrary signal, and their convolution in right. Notice that $X(j\Omega)$ is not symmetric, which means that $x(t)$ is complex-valued.

38. **Problem:** Suppose that a continuous-time signal $x(t)$ and its spectrum $|X(j\Omega)|$ in Figure 67 are known. The highest frequency component in the signal is f_h. The signal is sampled with frequency f_s, i.e. the interval between samples is $T_s = 1/f_s$: $x[n] = x(nT_s)$. Sketch the spectrum $|X(e^{j\omega})|$ of the discrete-time signal, when (a) $f_h = 0.25\ f_s$, (b) $f_h = 0.5\ f_s$, (c) $f_h = 0.75\ f_s$.

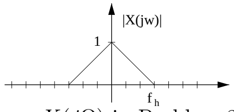


Figure 67: Spectrum $X(j\Omega)$ in Problem 38 also at page 14.

Solution: The spectrum $X(j\Omega)$ of a real analog signal is symmetric around y-axis. When sampling, the spectrum $X(e^{j\omega})$ is 2π-periodic (sampling frequency)

$$x[n] = x_p(nT_s) = x(t)p(t) \qquad \leftrightarrow \qquad X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\Omega - k\Omega_s))$$

a) Figure 68. The highest component of $x(t)$ is only $0.25 \cdot f_s \Rightarrow$ No aliasing.

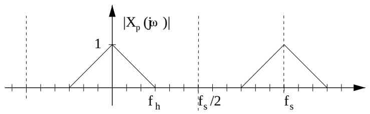


Figure 68: $f_h = 0.25\, f_s$, no aliasing in Problem 38(a).

b) Figure 69. Case: Nyquist frequency, half of the sampling frequency.

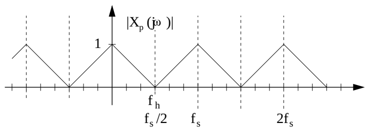


Figure 69: $f_h = 0.5\, f_s$, critical sampling in Problem 38(b).

c) Figure 70. Aliasing takes place. $X(e^{j\omega})$ is the sum of **all** folded analog spectra. The spectrum $X(e^{j\omega})$ is depicted in Figure 70 with a thick continuous line.

$$X(e^{j\omega}) \;\; = \;\; \frac{1}{T_s}\Big(... + X(j(\Omega - \Omega_s)) + X(j\Omega) + X(j(\Omega + \Omega_s)) + ...\Big)$$

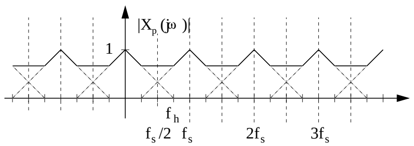


Figure 70: $f_h = 0.75\, f_s$, aliasing in Problem 38(c).

39. **Problem:** Consider a continuous-time signal $\tilde{x}(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)$, where f_1=100 Hz, f_2=300 Hz and f_3=750 Hz. The signal is sampled using frequency f_s. Sketch the magnitude of the Fourier spectrum of $x[n]$, when f_s equals to (i) 1600 Hz (ii) 800 Hz (iii) 400 Hz.

Use an ideal reconstruction lowpass filter whose cutoff frequency is $f_s/2$ for each case. What frequency components can be found in reconstructed analog signal $x_r(t)$?

Solution: There is a continuous-time signal

$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)$$

Let $f_1 = 100$ Hz, $f_2 = 300$ Hz and $f_3 = 750$ Hz.

It is possible directly to express the Fourier transform of a periodic signal using transform pairs (or see Page 82). In this case using Hertz

$$X(jf) = \pi \cdot \big(\delta[f + 750] + \delta[f + 300] + \delta[f + 100] + \delta[f - 100] + \delta[f - 300] + \delta[f - 750]\big)$$

The signal is sampled with sampling frequency f_s, $(T = 1/f_s)$.

$$x[n] = x(nT) = x(\frac{n}{f_s}) = \left(\cos(2\pi\frac{f_1}{f_s}n) + \cos(2\pi\frac{f_2}{f_s}n) + \cos(2\pi\frac{f_3}{f_s}n)\right)$$

In the frequency domain the discrete-time spectrum $G_p(j\Omega)$ can be seen as a sum of shifted and scaled replicas of the analog spectrum $G_a(j\Omega)$ as shown in Problems 37 and 38 (*Mitra 2Ed Eq. 5.9, p. 302 / 3Ed Eq. 4.10, p. 174*):

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

Alternatively, sampling can be considered as flipping the analog spectrum around each half of the sampling frequency down to the band $0 \ldots f_s/2$.

Reconstruction means converting a digital sequence back to analog signal. An ideal lowpass filter with the passband up to half of the sampling frequency is used. When reconstructing signals we can only observe frequencies up to Nyquist frequency.[1] If there are frequencies over the Nyquist frequency in the original signal, those frequencies are aliased into low frequencies.

In this problem $X(j\Omega)$ is sampled with three different sampling frequencies f_s of 1600 Hz, 800 Hz and 400 Hz, The Nyquist frequency is the half of the sampling frequency $f_s/2$, 800 Hz, 400 Hz, and 200 Hz, respectively. Let f_m (in this case 750 Hz) be the biggest frequency found in the input signal. If the sampling frequency is less than $2f_m = 1500$ Hz, then there will be aliasing.

In the following figures for i, ii and iii, the scale and magnitude values for aliased frequencies are not exactly correct. Phase shifts in input signal cause that a pure addition of magnitudes will not hold. (The sum of two cosines with same frequency and phase shift of π is zero. However, in practice, this is rarely significant.)

i) $f_s = 1600$ Hz, highest frequency component $f_m = 750$ Hz. The inequality $1600 > 2 \cdot 750$ holds, hence, there is no aliasing. All three frequencies can be recovered. See Figure 71.

[1] There is variation in using "Nyqvist frequency" in the literature. It is either (1) half of the sampling frequency (*Mitra 2Ed p. 302 / 3Ed p. 174*) or (2) the highest frequency in the signal (*Mitra 2Ed p. 304 / 3Ed p. 176*). The first one is much more common. The reader should not confuse with this.

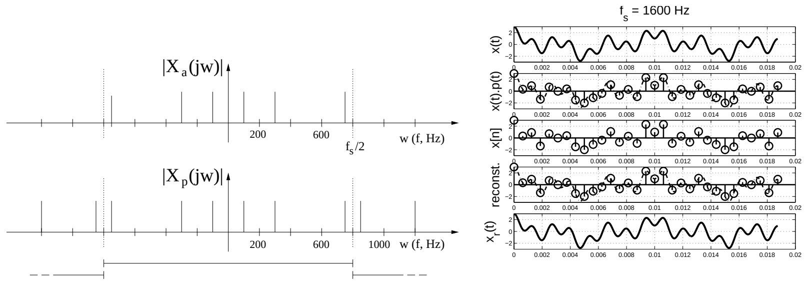


Figure 71: Sampling in Problem 39(i) with sampling frequency 1600 Hz: original analog spectrum $X(j\Omega)$ (left top), and spectrum $X(e^{j\omega})$ of the discrete-time signal (left bottom). Time domain view (right), top down $x(t)$, sampling $x(t) \cdot p(t)$ sampled sequence $x[n]$ to be processed with DSP, reconstruction, and reconstructed continuous-time signal $x_r(t)$. Again, in this case no aliasing, i.e. $x(t) \equiv x_r(t)$.

ii) $f_s = 800$ Hz, highest frequency component $f_m = 750$ Hz. The inequality $800 > 2 \cdot 750$ does not hold, hence, there is aliasing. All frequencies over 400 Hz are missed (750 Hz in this case); they cannot be observed. There is a new alias component at frequency 50 Hz. See Figure 72.

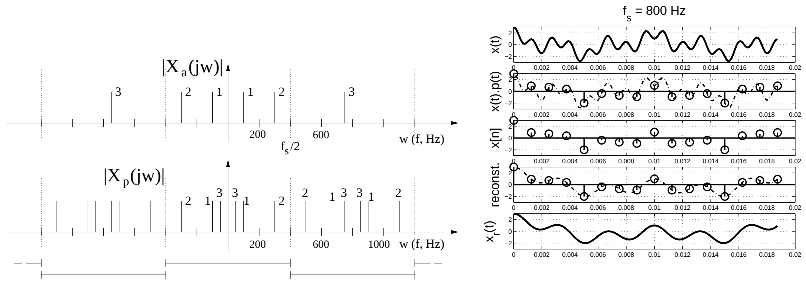


Figure 72: Sampling in Problem 39(ii) with sampling frequency 800 Hz. Aliasing occurs, $x(t) \neq x_r(t)$, compare the top and bottom axis in the figure right.

Before going further, there is a short demonstration on the aliasing signal component $x_3(t)$ ($f_i = 750$ Hz) of the signal $x(t)$ in Figure 73. The figures are depicted in time-domain: (a) original $x(t)$ with period $T = 1/f = (1/750) = 1.333$ ms, (b) samples $x[n]$ using interval $T_s = 1/f_s = (1/800) = 1.250$ ms, (c) reconstructed signal $x_{3r}(t)$, whose period $T_r = (1/50) = 20$ ms. The same aliasing effect can be shown using the cosine function, which is 2π-periodic ($\cos(\omega n) \equiv \cos(\omega n + 2\pi)$) and even ($\cos(-\omega n) \equiv \cos(\omega n)$). The highest component $x_3(t)$ of 750 Hz aliases in the sampling and reconstructing process to 50 Hz:

$$
\begin{aligned}
x_3(t) &= \cos(2\pi\ 750t) && \text{| original: 750 Hz} \\
x_3[n] = x_3(n/f_s) &= \cos(2\pi(750/800)n) = \cos(2\pi(750/800)n - 2\pi n) && \text{| } 2\pi\text{-periodicity} \\
&= \cos(2\pi(-50/800)n) = \cos(2\pi(50/800)n) && \text{| even function} \\
x_{3r}(t) &= \cos(2\pi\ 50t) && \text{| reconstructed: 50 Hz}
\end{aligned}
$$

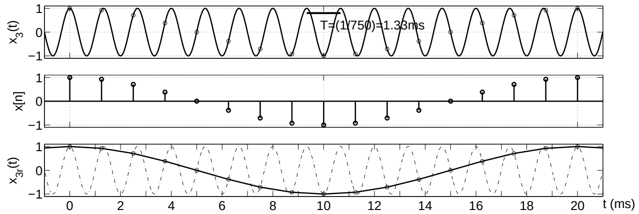


Figure 73: Demonstration of aliasing of a single cosine in Problem 39(ii).

iii) $f_s = 400$ Hz, highest frequency component $f_m = 750$ Hz. The inequality $400 > 2 \cdot 750$ does not hold, hence, there is aliasing. All frequencies over 200 Hz are missed (300 and 750 Hz). There are new alias components at frequencies 50 and 100 Hz. See Figure 74.
$\cos(2\pi\frac{750}{400}n) = \cos(2\pi\frac{750}{400}n - 4\pi n) = \cos(2\pi\frac{-50}{400}n) = \cos(2\pi\frac{50}{400}n)$
$\cos(2\pi\frac{300}{400}n) = \cos(2\pi\frac{300}{400}n - 2\pi n) = \cos(2\pi\frac{-100}{400}n) = \cos(2\pi\frac{100}{400}n)$

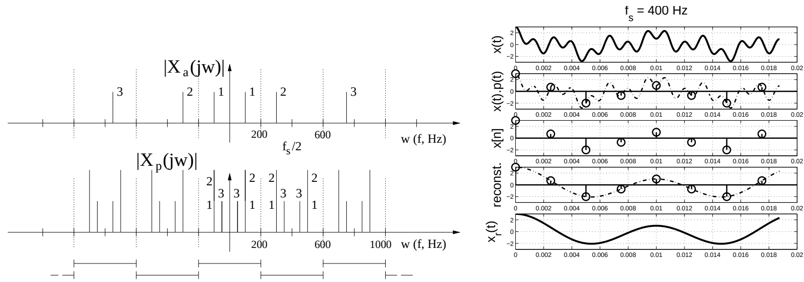


Figure 74: Sampling in Problem 39(iii) with sampling frequency 400 Hz. Aliasing occurs again, $x(t) \neq x_r(t)$.

After ideal reconstruction ($x[n] \to x_r(t)$) there are the following components left:

(i) original 100, 300, 750 Hz.

(ii) original 100, 300 Hz, and an alias 50 Hz.

(iii) original 100 Hz, and aliases 50, 100 Hz.

There is a sampling (aliasing) demo in the demo section in the course web pages `http://www.cis.hut.fi/Opinnot/T-61.3010/` Demo can also be loaded to Matlab.

40. **Problem:** Sketch specifications and compute the order for an anti-aliasing Butterworth filter with $f_s = 8$ kHz, interesting band $0 \ldots 2$ kHz, and minimum stopband attenuation 50 dB.

Solution: An anti-aliasing filter is an analog lowpass filter used in order to remove components, which cause aliasing when sampling (*Mitra 2Ed Sec. 5.6 / 3Ed Sec. 4.6*). Consider an analog signal $x(t)$ and its spectrum $X(j\Omega)$ depicted in Figure 75.

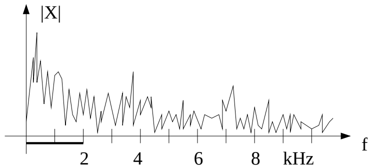


Figure 75: Spectrum $X(j\Omega)$ in Problem 40.

In the following, notations of (*Mitra 2Ed Fig. 5.28 / 3Ed Fig. 4.34*) are used, Ω_p for passband edge frequency, $\Omega_0 = \Omega_T - \Omega_p$ for stopband edge frequency, and Ω_T for sampling frequency. Now that the interesting band stops at $\Omega_p = 2$ kHz and the sampling frequency is $\Omega_T = 8$ kHz, we can set the edge frequency for the stopband to be at $\Omega_0 = (8-2) = 6$ kHz (see Figure 76). After sampling there will be aliasing components in $2 \ldots 4$ kHz, but we are not interested in them, i.e. we use digital filtering for that band.

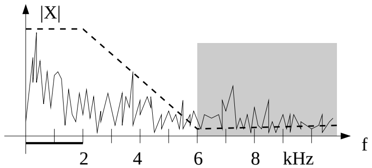


Figure 76: Problem 40: Spectrum $X(j\Omega)$, specifications for a LP filter (dashed line), frequency components that would alias in $0 \ldots 2$ kHz without anti-aliasing filtering (gray).

When the specifications are not so tight as they normally (cut-off at 4 kHz) are, also the order of the anti-aliasing filter is lower. The design of the anti-aliasing filter can be made even easier by increasing sampling frequency with analog circuits (order of anti-aliasing filter decreases), and afterwards decrease sampling frequency using multirate techniques (*Mitra 2Ed Sec. 10 / 3Ed Sec. 13*).

Calculations using (*Mitra 2Ed Table 5.1 / 3Ed Table 4.1*) or Table 7: $\Omega_0/\Omega_p = 3 \rightarrow N = \lceil 50/9.54 \rceil = 6$. Note that if the passband ended at 2 kHz and the stopband started at 4 kHz, the required order of the filter would be 10.

$\Omega_0 =$	$2\Omega_p$	$3\Omega_p$	$4\Omega_p$
Attenuation (dB)	$6.02N$	$9.54N$	$12.04N$
$\Omega_T =$	$3\Omega_p$	$4\Omega_p$	$5\Omega_p$

Table 7: Approximate minimum stopband attenuation of a Butterworth lowpass filter, (*Mitra 2Ed Table 5.1, p. 336 / 3Ed Table 4.1, p. 210*). See the text in Problem 40 for details.

41. **Problem:** Derive the transfer function of the feedback system shown in Figure 77.

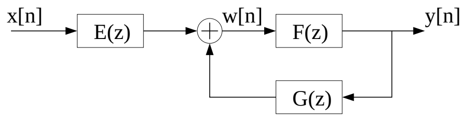


Figure 77: System in Problem 41.

Solution: Systems in parallel, see Figure 78: $H_p(z) = H_1(z) + H_2(z)$ in frequency domain and $h_p[n] = h_1[n] + h_2[n]$ in time domain. Systems in cascade, see Figure 79:

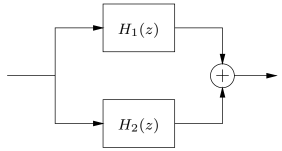


Figure 78: Systems in parallel, example in Problem 41.

$H_c(z) = H_1(z)H_2(z)$ in frequency domain and $h_c[n] = h_1[n] \circledast h_2[n]$ in time domain. The flow diagram of the system being investigated with temporary variable $w[n]$ is in

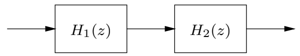


Figure 79: Systems in cascade, example in Problem 41.

Figure 80. We get the following equations:

$$\begin{cases} Y(z) &= F(z)W(z) \\ W(z) &= E(z)X(z) + G(z)Y(z) \end{cases}$$

$$\begin{aligned} Y(z) &= F(z)\left(E(z)X(z) + G(z)Y(z)\right) \\ Y(z)\left(1 - F(z)G(z)\right) &= \left(F(z)E(z)\right)X(z) \\ H(z) = \frac{Y(z)}{X(z)} &= \frac{F(z)E(z)}{1 - F(z)G(z)} \end{aligned}$$

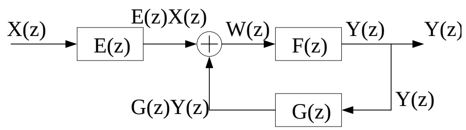


Figure 80: System in Problem 41.

42. **Problem:** Develop a polyphase realization of a length-9 FIR transfer function given by

$$H(z) = \sum_{n=0}^{8} h[n] z^{-n}$$

with (a) 2 branches and (b) 4 branches.

Solution: Polyphase realizations (*Mitra 2Ed Sec. 6.3.3 / 3Ed Sec. 8.3.3*) can be used in multirate techniques.

a) Two branches

$$\begin{aligned}
H(z) &= \sum_{n=0}^{8} h[n] z^{-n} \\
&= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + \\
&\quad h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8} \\
&= \left(h[0] + h[2]z^{-2} + h[4]z^{-4} + h[6]z^{-6} + h[8]z^{-8}\right) + \\
&\quad z^{-1}\left(h[1] + h[3]z^{-2} + h[5]z^{-4} + h[7]z^{-6}\right) \\
&= H_0(z^2) + z^{-1} H_1(z^2)
\end{aligned}$$

where

$$\begin{aligned}
H_0(z) &= h[0] + h[2]z^{-1} + h[4]z^{-2} + h[6]z^{-3} + h[8]z^{-4} \\
H_1(z) &= h[1] + h[3]z^{-1} + h[5]z^{-2} + h[7]z^{-3}
\end{aligned}$$

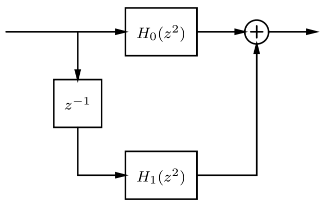


Figure 81: Polyphase realization with two branches in Problem 42(a).

b) Four branches

$$\begin{aligned}
H(z) &= \sum_{n=0}^{8} h[n] z^{-n} \\
&= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + \\
&\quad h[5]z^{-5} + h[6]z^{-6} + h[7]z^{-7} + h[8]z^{-8} \\
&= \left(h[0] + h[4]z^{-4} + h[8]z^{-8}\right) + z^{-1}\left(h[1] + h[5]z^{-4}\right) + \\
&\quad z^{-2}\left(h[2] + h[6]z^{-4}\right) + z^{-3}\left(h[3] + h[7]z^{-4}\right) \\
&= H_0(z^4) + z^{-1}H_1(z^4) + z^{-2}H_2(z^4) + z^{-3}H_3(z^4)
\end{aligned}$$

where

$$\begin{aligned}
H_0(z) &= h[0] + h[4]z^{-1} + h[8]z^{-2} \\
H_1(z) &= h[1] + h[5]z^{-1} \\
H_2(z) &= h[2] + h[6]z^{-1} \\
H_3(z) &= h[3] + h[7]z^{-1}
\end{aligned}$$

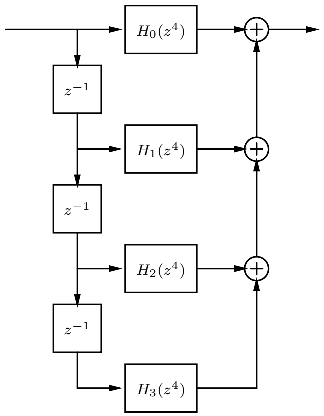


Figure 82: Polyphase realization with four branches in Problem 42(b).

43. **Problem:** Analyze the digital filter structure shown in Figure 83 and determine its transfer function $H(z) = Y(z)/X(z)$.

a) Is the system LTI?

b) Is the structure canonic to delays?

c) Compute $H(z)H(z^{-1})$ (the squared amplitude response). What is the type of this filter (lowpass/highpass/bandpass/bandstop/allpass)?

Solution: Let us use three temporary signals $w_1[n]$, $w_2[n]$, and $w_3[n]$, in the following locations in Figure 83.

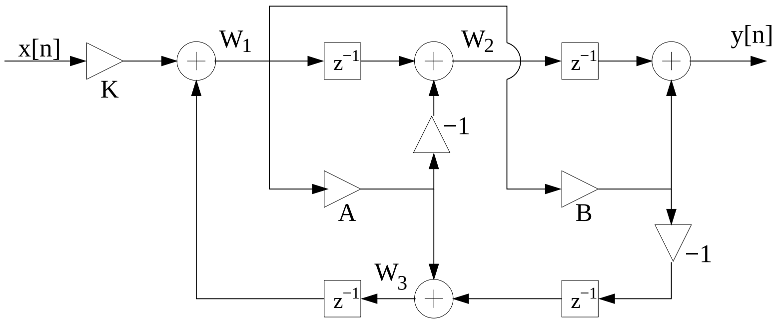


Figure 83: The filter with temporary signals w_1, w_2, and w_3 in Problem 43.

From the figure we get the following expressions in transform-domain ($W \equiv W(z)$):

$$\begin{aligned} W_1 &= KX + z^{-1}W_3 \\ W_2 &= (z^{-1} - A)W_1 \\ W_3 &= AW_1 - Bz^{-1}W_1 = (A - Bz^{-1})W_1 \\ Y &= z^{-1}W_2 + BW_1 \end{aligned}$$

Substituting the equation from third line to first line we get

$$\begin{aligned} W_1 &= KX + z^{-1}(A - Bz^{-1})W_1 \\ (1 - Az^{-1} + Bz^{-2})W_1 &= KX \end{aligned}$$

Next, substituting second line in fourth line we get

$$Y = [z^{-1}(z^{-1} - A) + B]W_1$$

Finally, we get rid of the last temporary variable W_1, and get

$$H(z) = \frac{Y(z)}{X(z)} = K \cdot \frac{B - Az^{-1} + z^{-2}}{1 - Az^{-1} + Bz^{-2}}$$

a) It is LTI. There are only multiplications by constants, delays, and sums of sequences.

b) Since the structure employs 4 unit delays to implement a second-order transfer function, it is not canonic.

Canonic structure: the number of registers, i.e. delay components, is the same as the filter order. Direct form I is not canonic, but it is intuitive and its difference equation is easy to obtain. Direct form II is canonic. It is more efficient to use canonic structures. (Consider, for example, Problem 51. If canonic structure is used, there are only 8 storage locations instead of 10.)

c)

$$\begin{aligned} H(z)H(z^{-1}) &= K^2 \left(\frac{B - Az^{-1} + z^{-2}}{1 - Az^{-1} + Bz^{-2}} \right) \left(\frac{B - Az^{1} + z^{2}}{1 - Az^{1} + Bz^{2}} \right) \quad | \cdot \frac{z^{-2}}{z^{-2}} \\ &= K^2 \left(\frac{B - Az^{-1} + z^{-2}}{1 - Az^{-1} + Bz^{-2}} \right) \left(\frac{Bz^{-2} - Az^{-1} + 1}{z^{-2} - Az^{-1} + B} \right) \\ &= K^2 \end{aligned}$$

Therefore $|H(e^{j\omega})| = K$ for all values of ω and hence $|H(e^{j\omega})| = 1$ if $K = 1$. $H(z)$ is an allpass transfer if $K = 1$.

44. **Problem:** The filter in Figure 84 is in canonic direct form II (DF II). Draw it in DF I. What is the transfer function $H(z)$?

Solution: Direct form structure means that the coefficients of the block diagram are the same (or negative values) as in the difference equation and transfer function. There are also other structures, e.g. lattice. The transfer function for any direct form (I, II, and transposes I_T, II_T, respectively, see Page 94) is the same. Some differences (may) occur when working with finite word length. There are also differences in computational load and memory storage.

a) The block diagram in Figure 84 is in canonic direct form II.

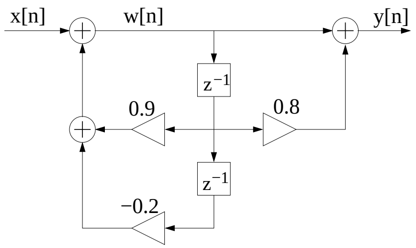


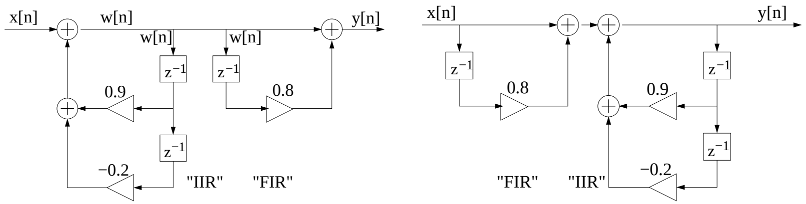
Figure 84: The block diagram of direct form II in Problem 44.

If we want to convert it into direct form I without any calculations (done below in (b)), we can duplicate the registers. The same signal $w[n]$ goes into the both branches. See Figure 85(a).

Then we can denote the part in left as an "IIR subsystem" and the structure in right as an "FIR subsystem". Because both of them are LTI, we can change the order of them, as in any LTI system, for example, using impulse responses

$$h[n] = h_{IIR}[n] \circledast h_{FIR}[n] \equiv h_{FIR}[n] \circledast h_{IIR}[n]$$

Now we have direct form I in Figure 85(b), and the difference equation and the transfer function can be obtained directly without any temporal variables! However, there are now three registers instead of two.



(a) (b)

Figure 85: From direct form II to direct form I in Problem 44(a).

b) The transfer function and difference equation can be derived directly from the filter in Figure 84:

$$\begin{aligned} y[n] &= w[n] + 0.8w[n-1] \\ w[n] &= x[n] + 0.9w[n-1] - 0.2w[n-2] \end{aligned}$$

Using z-transform

$$\begin{aligned} Y(z) &= W(z) + 0.8z^{-1}W(z) = W(z)(1 + 0.8z^{-1}) \\ W(z) &= X(z) + 0.9z^{-1}W(z) - 0.2z^{-2}W(z) \end{aligned}$$

From the latter one, $W(z) = X(z)/(1 - 0.9z^{-1} + 0.2^{-2})$, and substituting into the first one, we get

$$\begin{aligned} Y(z) &= X(z)\frac{1 + 0.8z^{-1}}{1 - 0.9z^{-1} + 0.2^{-2}} \\ H(z) = Y(z)/X(z) &= \frac{1 + 0.8z^{-1}}{1 - 0.9z^{-1} + 0.2^{-2}} \end{aligned}$$

Using inverse z-transform we get difference equation which can be easily drawn as direct form I block diagram:

$$\begin{aligned} Y(z)/X(z) &= \frac{1 + 0.8z^{-1}}{1 - 0.9z^{-1} + 0.2^{-2}} \\ Y(z)(1 - 0.9z^{-1} + 0.2^{-2}) &= X(z)(1 + 0.8z^{-1}) \\ y[n] - 0.9y[n-1] + 0.2y[n-2] &= x[n] + 0.8x[n-1] \end{aligned}$$

Remark. Direct Forms.

(*Mitra 2Ed Sec. 6.4.1 / 3Ed Sec. 8.4.1*) Direct form: coefficients of difference equation or transfer function can be found in block diagram. (This is not the case, for example, in lattice form.) Common in all forms is that they have the same transfer function, but the "implementation" is different.

Let the transfer function be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.5z^{-1}}{1 - 0.2z^{-1} + 0.4z^{-2}}$$

In the top numerator polynomial $1 + 0.5z^{-1}$ refers to "FIR part" $P(z)$ and in the bottom denominator polynomial $1 - 0.2z^{-1} + 0.4z^{-2}$ "IIR part" $D(z)$:

$$H(z) = P(z)\frac{1}{D(z)}$$

How to get difference equation and block diagram from tranfer function,
z-transform $ax[n - n_0] \leftrightarrow a\, z^{-n_0}\, X(e^{j\omega})$:

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{1 + 0.5z^{-1}}{1 - 0.2z^{-1} + 0.4z^{-2}} \\ Y(z) &= \frac{X(z)[1 + 0.5z^{-1}]}{1 - 0.2z^{-1} + 0.4z^{-2}} \\ Y(z)[1 - 0.2z^{-1} + 0.4z^{-2}] &= X(z)[1 + 0.5z^{-1}] \\ Y(z) - 0.2z^{-1}Y(z) + 0.4z^{-2}Y(z) &= X(z) + 0.5z^{-1}X(z) \\ y[n] - 0.2y[n-1] + 0.4y[n-2] &= x[n] + 0.5x[n-1] \\ y[n] &= 0.2y[n-1] - 0.4y[n-2] + x[n] + 0.5x[n-1] \end{aligned}$$

Direct form I can be drawn directly $H(z) = P(z) \cdot \frac{1}{D(z)}$, first "FIR" and then "IIR" (Figure 86).

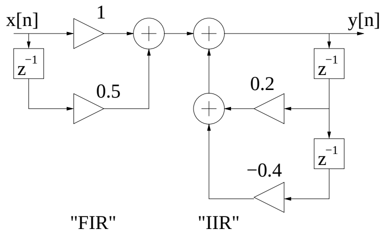
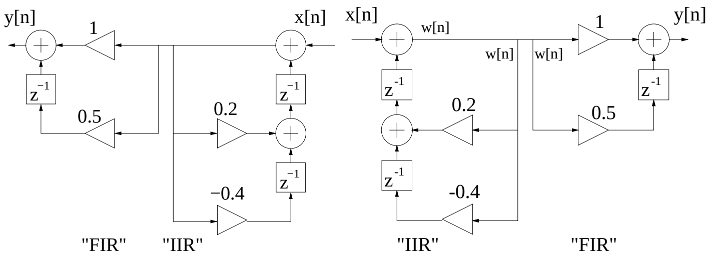


Figure 86: Direct form I. You may connect FIR and IIR parts in the middle sum line.

When transposing (Figure 87) transfer function stays, but structure changes. "Rules" for transposing:

1 Change directions
2 Nodes to sums
3 Sums to nodes
4 Flip the whole structure



Phases 1-3

Phase 4: flip around. You may connect parts in the middle line.

Figure 87: Transposed direct form I.

Direct form II contains minimum number of delay registers. Draw in order "IIR" and then "FIR". Think the transfer function in order $H(z) = \frac{1}{D(z)} \cdot P(z)$. Because LTI, the order of subfilters can be changed. Connect the delay registers, because there are the same signals (see Book). So you get **canonic** form, where the number of delays is the same as order of the filter (Figure 88).

Corresponding transponing II_T, see Figure 89.

Example on direct form, cascade and parallel system. Consider a second order transfer function

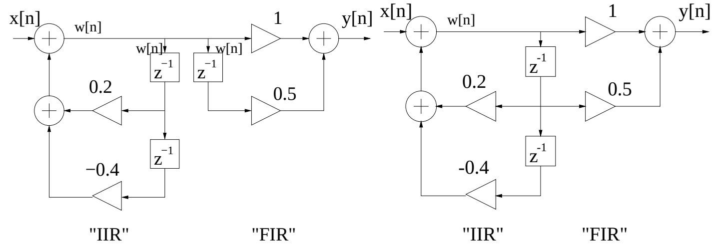
$$H(z) = \frac{1}{(1+\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{1}{1+\frac{1}{12}z^{-1}-\frac{1}{12}z^{-2}}$$

with difference equation

$$y[n] = -\frac{1}{12}y[n-1] + \frac{1}{12}y[n-2] + x[n]$$

Cascade form can be written as

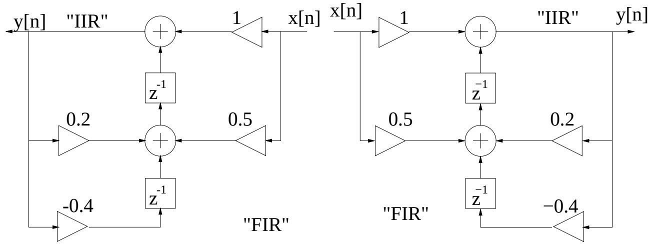
$$H(z) = \left(\frac{1}{1+\frac{1}{3}z^{-1}}\right)\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)$$



IIR-FIR before connecting

Canonic form

Figure 88: Direct form II.



Phases 1-3

Phase 4: flip around

Figure 89: Transposed direct form II.

and parallel form using partial fraction (draw!)

$$H(z) = \frac{\frac{4}{7}}{1+\frac{1}{3}z^{-1}} + \frac{\frac{3}{7}}{1-\frac{1}{4}z^{-1}}$$

45. **Problem:** Develop a canonic direct form realization of the transfer function

$$H(z) = \frac{2 + 4z^{-1} - 7z^{-2} + 3z^{-5}}{1 + 2z^{-1} + 5z^{-3}}$$

and then determine its transpose configuration.

Solution: There is a canonic direct form II realization of $H(z)$ in Figure 90. Its transposed realization can be achieved

- by changing the direction of the flow to opposite,
- by replacing each sum node with a branch node, and
- by replacing each branch node with a sum node

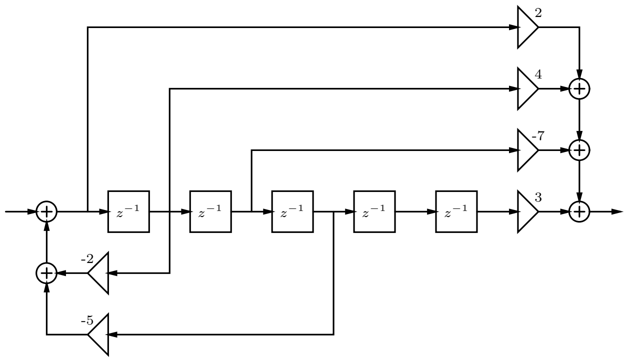


Figure 90: Canonic direct form II in Problem 45.

The end result is in Figure 91.

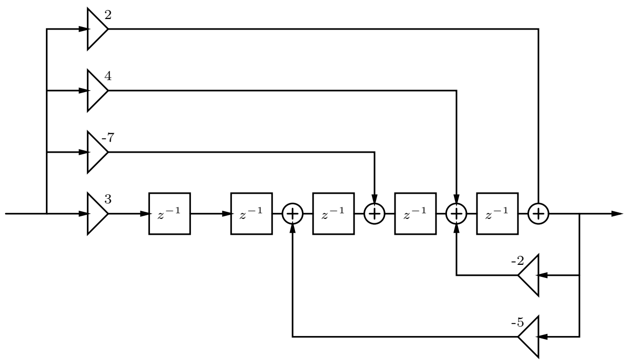


Figure 91: Transposed canonic direct form II in Problem 45.

46. **Problem:** Sketch the following specifications of a digital filter on paper. Which of the amplitude responses of the realizations in Figure 92 do fulfill the specifications?

Specifications: Digital lowpass filter, sampling frequency f_T 8000 Hz, passband edge frequency f_p 1000 Hz, transition band 500 Hz (transition band is the band between passband and stopband edge frequencies!), maximum passband attenuation 3 dB, minimum stopband attenuation 40 dB.

Solution: The frequency specifications are in Hertz, radians, and in normalized Matlab frequency in Table 8 and they are drawn in Figure 92 with dashed line.

sampling frequency	f_T	8000 Hz	ω_T	2π (rad)		2
passband edge	f_p	1000 Hz	ω_p	$\pi/4$ (rad)	`Wp`	$2 \cdot 1000/8000 = 0.25$
stopband edge	f_s	1500 Hz	ω_s	$3\pi/8$ (rad)	`Ws`	$2 \cdot 1500/8000 = 0.375$
passband ripple	Rp	3 dB			`Rp`	3
stopband attenuation	Rs	40 dB			`Rs`	40

Table 8: Specifications for the filter in Problem 46.

Now that specifications are written and sketched, the filter order and the filter coefficients are computed using a specific software (e.g. Matlab, `ellipord` and `ellip`, `buttord` and `butter`, etc.). Then the amplitude response $|H(e^{j\omega})|$ of the calculated filter is plotted in the same picture as the sketch of the specifications (e.g. Matlab, `[...] = freqz(B,A,...);`). If the amplitude response curve fits in the specifications, we have succeeded. In other case, the specifications and the code for the filter are re-checked.

The elliptic IIR filter in Figure 92(a) (via bilinear transform) is of order 4 and it fulfills the specifications exactly.

Chebychev II filter (Figure 92(b)), which is 10th order IIR, is monotonic in passband and has stopband attenuation of 50 dB instead of 40. The amplitude response fits in the allowed area, and it is already too strict. Probably the order $N = 8$ would be sufficient.

The third filter (Figure 92(c)) is 50th order FIR, whose transition is narrow enough but at the wrong cut-off frequency. So, this is the only filter, which does not fulfill the specifications. One should check the cut-off frequency so that the amplitude response fits.

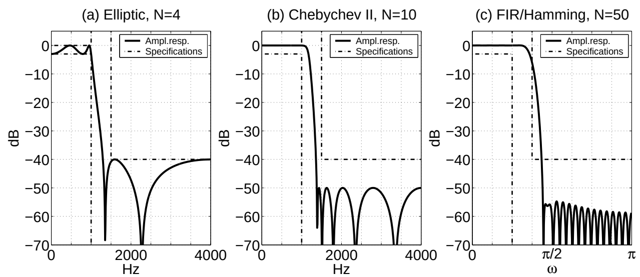


Figure 92: Three realizations in Problem 46: amplitude responses (solid line) with specifications (dashed line) of (a) 4th order elliptic (OK!), (b) 10th order Chebychev II (OK, too tight realization?), (c) 50th order FIR using Hamming window (bad cut-off frequency).

47. **Problem:** Connect first each amplitude response to the corresponding pole-zero plot in Figure 93. Then recognize the following digital IIR filter algoritms: Butterworth, Chebyshev I, Chebyshev II, Elliptic. The conversion from analog to digital form is done using bilinear transform.

Solution: Analog filter design is represented in (*Mitra 2Ed Sec. 5.4 / 3Ed Sec. 4.4*). The approximations are given with magnitude-squared responses of Nth order in Table 9.

Approximation	M 2Ed Sec.	M 3Ed Sec.	Response
Butterworth	5.4.2	4.4.2	$\lvert H_a(j\Omega)\rvert^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$
Chebyshev I	5.4.3	4.4.3	$\lvert H_a(j\Omega)\rvert^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_p)}$
Chebyshev II	5.4.3	4.4.3	$\lvert H_a(j\Omega)\rvert^2 = \frac{1}{1+\epsilon^2[\frac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}]^2}$
Elliptic	5.4.4	4.4.4	$\lvert H_a(j\Omega)\rvert^2 = \frac{1}{1+\epsilon^2 R_N^2(\Omega/\Omega_p)}$

Table 9: Analog filter approximations in Problem 47.

The response of Butterworth is monotonic. Chebyshev I is equiripple in the passband and monotonic in the stopband whereas Chebyshev II is monotonic in the passband and equiripple in the stopband. Elliptic approximation is equiripple both in the passband and stopband. The filter order can often be obtained by computing the number of local maximum and minimum.

The digital filters are obtained through bilinear transform (*Mitra 2Ed Sec. 7.2 / 3Ed Sec. 9.2*). Hence, approximations, amplitude responses and pole-zero plots are related to each other according to the Figure 93.

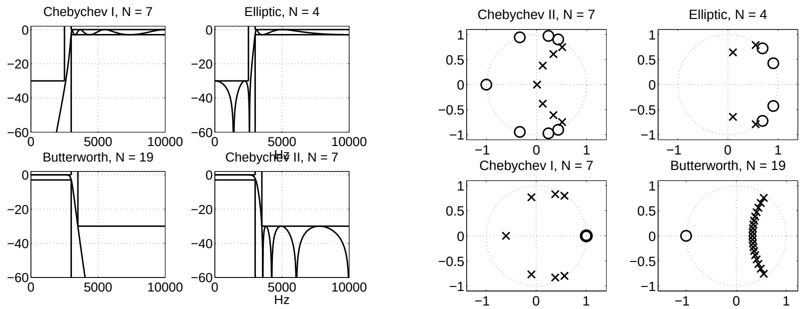


Figure 93: Problem 47, see the titles of each subfigure for filter type and order.

48. **Problem:** Consider the following digital **lowpass** filter of type Chebyshev II:

$$H(z) = K \cdot \frac{0.71 - 0.36z^{-1} - 0.36z^{-2} + 0.71z^{-3}}{1 - 2.11z^{-1} + 1.58z^{-2} - 0.40z^{-3}}$$

Normalize the maximum of the amplitude response to the unity (0 dB).

Solution: Chebyshev II approximation is monotonic in the passband, see Figure 94.

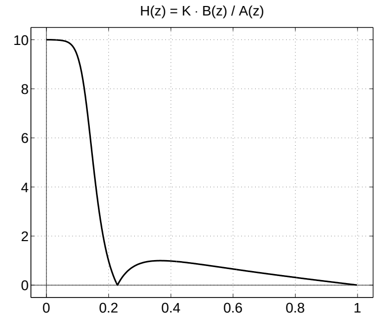


Figure 94: Problem 48, $H(z) = K \cdot B(z)/A(z)$ without magnitude scaling.

Therefore the maximum value of the amplitude response of the lowpass Chebyshev II filter is at $\omega = 0$. The gain K can be computed also in z-plane using $z = e^{j\omega}|_{\omega=0} = 1$.

$$\begin{aligned}
|H(z)| &= \left| K \frac{0.71 - 0.36z^{-1} - 0.36z^{-2} + 0.71z^{-3}}{1 - 2.11z^{-1} + 1.58z^{-2} - 0.40z^{-3}} \right| \\
|H(z)|_{z=1} &= \left| K \frac{0.71 - 0.36z^{-1} - 0.36z^{-2} + 0.71z^{-3}}{1 - 2.11z^{-1} + 1.58z^{-2} - 0.40z^{-3}} \right| = 1 \\
&= K \frac{0.70}{0.07} = 1 \\
\Rightarrow K &= 0.1
\end{aligned}$$

Remember also that when $|H(z)|_{max} = 1$, then this maximum reference level is in (power) desibels $|H(z)|_{max} = 20\,\log_{10}(1) = 0$ dB.

49. **Problem:** Consider the following prototype analog Butterworth-type lowpass filter

$$H_{protoLP}(s) = \frac{1}{s+1}$$

a) Form an analog first-order lowpass filter with cutoff frequency Ω_c by substituting $H(s) = H_{protoLP}(\frac{s}{\Omega_c})$. Draw the pole-zero plot in s-plane.

b) Implement a discrete first-order lowpass filter $H_{Imp}(z)$, whose cutoff frequency (-3 dB) is at $f_c = 100$ Hz and sampling rate is $f_s = 1000$ Hz, applying the impulse-invariant method to $H(s)$. Draw the pole-zero plot of the filter $H_{Imp}(z)$.

c) Implement a discrete first-order lowpass filter $H_{Bil}(z)$ with the same specifications applying the bilinear transform to $H(s)$. Prewarp the edge frequency. Draw the pole-zero plot of the filter $H_{Bil}(z)$.

Solution: The solution to the problem starts from the page 102. Two methods for digital IIR design are shown in the lecture slides, impulse invariant method and bilinear transform method.

Analog Butterworth lowpass filter

Analog Butterworth filter is discussed in (*Mitra 2Ed Sec. 5.4.2 / 3Ed Sec. 4.4.2*). The definition of an analog Butterworth filter with cut-off frequency Ω_c is $|H_a(j\Omega)|^2 = 1/(1+(\frac{\Omega}{\Omega_c})^{2N})$ (*Mitra 2Ed Eq. 5.31 / 3Ed Eq. 4.33*). The first order ($N=1$) filter is therefore

$$\begin{aligned}
|H_a(j\Omega)|^2 &= \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^2} \\
H_a(s)H_a(-s) &= \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^2} = \frac{1}{1-\left(\frac{s}{\Omega_c}\right)^2} \\
&= \overbrace{\frac{1}{1+\left(\frac{s}{\Omega_c}\right)}}^{=H(s)} \cdot \overbrace{\frac{1}{1+\left(\frac{-s}{\Omega_c}\right)}}^{H(-s)}
\end{aligned}$$

where $s = j\Omega$

$$\Rightarrow H_a(s) = \frac{\Omega_c}{s+\Omega_c}$$

The pole in s-plane is at $s = -\Omega_c$.

Here, Ω refers to frequency in analog domain ($H(j\Omega)$) and ω to frequency in digital domain ($H(e^{j\omega})$).

As said earlier, there are two ways to convert analog filter to digital. The inpulse-invariant method is straigtforward but it has severe limitations. The bilinear transform is a standard way.

Impulse-invariant method, see, e.g. lecture slides:

$$H_a(s) \mapsto h_a(t) \mapsto h[n] = h_a(nT) \mapsto H(z)$$

In the impulse-invariant method the target is to get impulse response of digital filter $h[n]$ to be the same as the sampled impulse response of analog filter $h_a(nT)$. Because IIR filters have normally an impulse response of infinite length, this method brings distortion.

The **bilinear transformation** is acquired when

$$s = k \cdot \frac{1-z^{-1}}{1+z^{-1}}$$

is inserted into the system function (*Mitra 2Ed Eq. 7.21 / 3Ed Eq. 9.15*)

$$H(z) = H_a(s)\big|_{s=k\cdot\frac{1-z^{-1}}{1+z^{-1}}}$$

Note that here k is a parameter used in the derivation of the bilinear transformation. It is originally $k = (2/T)$ but can be set $k = 1$ to simplify the procedure.

The **frequency is warped** before the bilinear transformation (*Mitra 2Ed Fig. 7.4, 7.5 / 3Ed Fig. 9.3, 9.4*). In the small frequencies the difference is not big, but it is significant in high frequencies. Therefore the discrete-time normalized angular cut-off frequency ω_c has to be first **prewarped** into analog-time prewarped cut-off frequency Ω_{pc}:

$$\Omega_{pc} = k \cdot \tan\left(\frac{\omega_c}{2}\right)$$

where $\omega_c = 2\pi f_c/f_T = 2\pi f_c T = \Omega_c T$, and $0 < \omega_c < \pi$, and $[f_c] =$ Hz, and $f_T = 1/T$ is the sampling frequency. For example, if discrete-time $f_c = 100$ Hz and $f_s = 1000$ Hz, then $\Omega_{pc} = 2000 \cdot \tan(0.1\pi)$, and $f_{pc} \approx 103.4$ Hz. Analog design has to be done using f_{pc} instead of f_c in order to get the cut-off frequency to 100 Hz in the digital filter.

Solution to Problem 49

a) Substitution gives directly

$$H(s) = H_{protoLP}(s/\Omega_c) = \frac{\Omega_c}{s+\Omega_c}$$

The pole-zero plot of a lowpass filter in s-plane is in Figure 95.

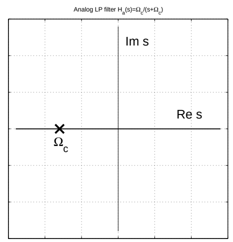


Figure 95: Problem 49(a), LP in s-plane. The stable pole is at $s = -\Omega_c$ in the left subspace, the y-axis is the frequency.

b) Transfer function using the impulse-invariant method.

$$H_a(s) = \frac{\Omega_c}{s+\Omega_c} \mapsto h_a(t) = \Omega_c e^{-\Omega_c t}\mu(t) \mapsto$$

$$h[n] = h_a(nT) = \Omega_c e^{-\Omega_c nT}\mu[n] \mapsto H(z) = \Omega_c \sum_{n=0}^{\infty} e^{-\Omega_c nT} z^{-n} = \frac{\Omega_c}{1-e^{-\Omega_c T}z^{-1}}$$

The constant K is introduced in order to scale the maximum of $|H(e^{j\omega})|$ into unity. Using (*Mitra 2Ed Eq. 7.7 / 3Ed Eq. 9.7*), $\omega_c = \Omega_c/f_T = 2\pi f_c/f_T$ and values $f_T = 1$ kHz (sampling frequency) and $f_c = 100$ Hz (cut-off frequency),

$$H(z)_{Imp} = \frac{K}{1 - e^{-\omega_c} z^{-1}} = \frac{K}{1 - e^{-\pi/5} z^{-1}}$$

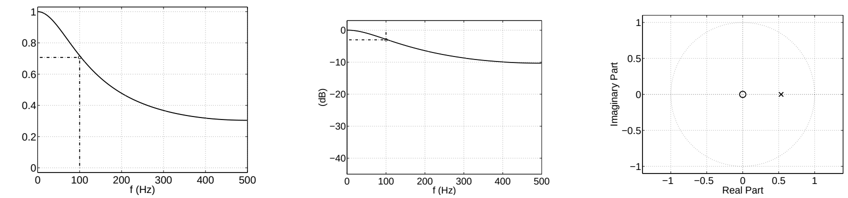
We also know that the maximum is located at zero frequency, because the frequency response of a Butterworth filter is monotonic. Thus we get

$$\frac{K}{1 - e^{-\pi/5}} = 1 \Leftrightarrow K = 1 - e^{-\pi/5}$$

The transfer function of the filter is therefore

$$H(z)_{Imp} = \frac{1 - e^{-\pi/5}}{1 - e^{-\pi/5} z^{-1}} = 0.4665 \cdot \frac{1}{1 - 0.5335 z^{-1}}$$

There is a pole at $z = 0.5335$, see Figure 96 for the amplitude response in linear scale, in desibels and the pole-zero plot.



(a) (b) (c)

Figure 96: Problem 49, the filter $H_{Imp}(z)$ using impulse-invariant method. (a) Amplitude response in linear scale $|H(e^{j\omega})|$ and (b) in desibels $10 \cdot \log_{10} |H(e^{j\omega})|^2$, (c) pole-zero diagram.

c) Transfer function using bilinear transform. Compute the normalized angular discrete-time cut-off frequency ω_c,

$$\omega_c = \frac{2\pi\Omega_c}{\Omega_s} = \frac{2\pi 2\pi f_c}{2\pi f_T} = \frac{2\pi f_c}{f_T} = 0.2\pi$$

and the prewarped cut-off frequency Ω_{pc} ($k = 2/T$):

$$\Omega_{pc} = k \cdot \tan(\frac{\omega_c}{2}) = k \cdot \tan(0.1\pi)$$

The digital filter is obtained through bilinear transform:

$$\begin{aligned} H(z) &= H(s)\big|_{s=k\cdot\frac{1-z^{-1}}{1+z^{-1}},\ \ \Omega_c=\Omega_{pc}=k\cdot\tan(0.1\pi)} \\ &= \frac{\Omega_c}{s+\Omega_c}\Big|_{s=k\cdot\frac{1-z^{-1}}{1+z^{-1}},\ \ \Omega_c=\Omega_{pc}=k\cdot\tan(0.1\pi)} \\ &= \frac{k \cdot \tan(0.1\pi)}{k \cdot \frac{1-z^{-1}}{1+z^{-1}} + k \cdot \tan(0.1\pi)} \qquad | \ \cancel{k} \\ &= \frac{\tan(0.1\pi)(1+z^{-1})}{(1+\tan(0.1\pi)) - (1-\tan(0.1\pi))z^{-1}} \end{aligned}$$

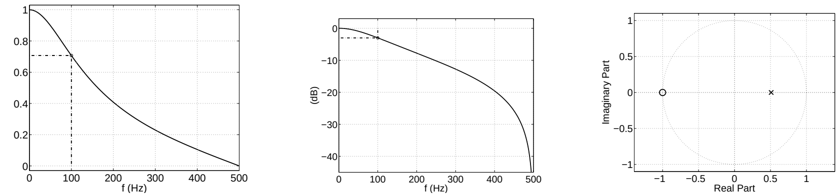
The last task is to normalize the transfer function. The constant term in denominator polynomial should be scaled to 1, and the maximum value of the amplitude response to 1. While this is a Butterworth lowpass filter, the maximum is reached at $\omega = 0$, i.e., $z = e^{j\omega}|_{\omega=0} = 1$.

$$|H(z)_{Bil}|_{max} = \left| K \cdot \frac{1+z^{-1}}{1 - \frac{1-\tan(0.1\pi)}{1+\tan(0.1\pi)} z^{-1}} \right|_{z=1} = 1$$

Finally,

$$H_{Bil}(z) = 0.2452 \cdot \frac{1+z^{-1}}{1 - 0.5095 z^{-1}}$$

There is a zero at $z = -1$ and a pole at $z = 0.5095$. See Figure 97 for the amplitude response in linear scale, in (power) desibels (`20*log10(A)=10*log10(A^2)`), and the pole-zero plot. Compare also to the filter obtained through the impulse-invariant method in Figure 96.



(a) (b) (c)

Figure 97: Problem 49, the filter $H_{Bil}(z)$ using bilinear transform. (a) Amplitude response in linear scale and (b) in desibels, (c) pole-zero diagram.

50. **Problem:** Use windowed Fourier series method and design a FIR-type (causal) lowpass filter with cutoff frequency $3\pi/4$. Let the order of the filter be 4.

a) Use the rectangular window of length 5.

b) Use the Hamming window of length 5.

c) Compare how the amplitude responses of the filters designed in (a) and (b) differ assuming that the window size is high enough (e.g. $M = 50$).

Solution: Digital FIR filter design with windowed (truncated) Fourier series method. The idea is to find infinite-length impulse response of the ideal filter and truncate it so that a realizable finite-length filter is obtained.

$$h_t[n] = h_d[n] \cdot w[n] \quad \leftrightarrow \quad H_t(z) = H_d(z) \circledast W(z)$$

Now, when cut-off frequency (-3 dB) is at $\omega_c = 3\pi/4$, the infinite-length impulse response of the ideal filter is:

$$h_d[n] = \sin(\frac{3\pi}{4}n)/(\pi n) = (3/4)\,\mathrm{sinc}(\frac{3}{4}n)$$

When computing values, $\sin(x)/x \to 1$, when $x \to 0$, or $\mathrm{sinc}(x) \to 1$, when $x \to 0$. So, we get $h_d[n] = \{\ldots, -0.1592, 0.2251, \underline{0.75}, 0.2251, -0.1592, \ldots\}$.

a) Now we are using rectangular window $w_r[n]$ of length 5 (4th order),

$$w_r[n] = \begin{cases} 1, & -2 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$h_t[n] = h_d[n] \cdot w_r[n] = \{-0.1592, 0.2251, \underline{0.75}, 0.2251, -0.1592\}$$

If causal filter is needed, then the shift by two is needed
$h_c[n] = h_t[n-2] = \{\underline{-0.1592}, 0.2251, 0.75, 0.2251, -0.1592\}$.
In Figure 98 time-domain view:
(a) $h_d[n]$ (IIR), (b) $w_r[n]$, and (c) $h_t[n] = h_d[n] \cdot w_r[n]$ (FIR).
In Figure 99 the corresponding frequency-domain view:
(a) $H_d(e^{j\omega})$ (ideal, desired), (b) $W_r(e^{j\omega})$, and (c) $H_t(e^{j\omega}) = H_d(e^{j\omega}) \circledast W_r(e^{j\omega})$ (realisable).

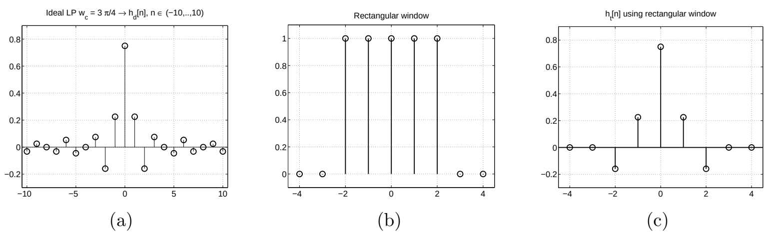


Figure 98: Problem 50(a): time domain view, (a) $h_d[n]$, (b) $w_r[n]$,(c) $h_t[n]$.

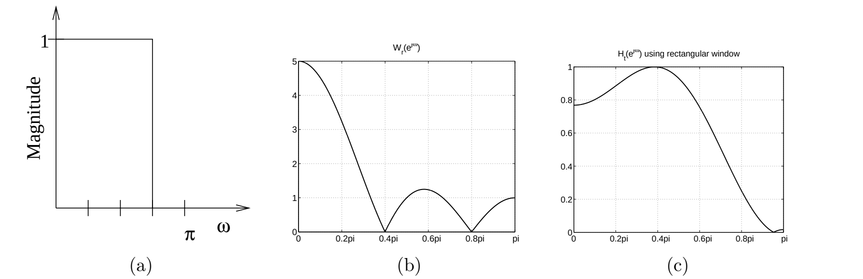


Figure 99: Problem 50(a): frequency domain $(0\ldots\pi)$, (a) $H_d(e^{j\omega})$, (b) $W_r(e^{j\omega})$, (c) $H_t(e^{j\omega})$.

b) Now we are using Hamming window[2] $w_h[n]$ of length 5,

$$w_h[n] = \begin{cases} 0.54 + 0.46\cos(2\pi n/4), & -2 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Hence,

$$\begin{aligned} h_t[n] &= h_d[n] \cdot w_h[n] = h_d[n] \cdot \big(0.54 + 0.46\cos(2\pi n/(2M))\big) \\ &= \{0.08 \cdot (-0.1592), 0.54 \cdot 0.2251, \underline{0.75}, 0.54 \cdot 0.2251, 0.08 \cdot (-0.15592)\} \\ &= \{-0.0127, 0.1215, \underline{0.75}, 0.1215, -0.0127\} \end{aligned}$$

If causal filter is needed, then
$h_c[n] = h_t[n-2] = \{\underline{-0.0127}, 0.1215, 0.75, 0.1215, -0.0127\}$
In Figure 100 time-domain view:
(a) $h_d[n]$, (b) $w_h[n]$, and (c) $h_t[n] = h_d[n] \cdot w_h[n]$.
In Figure 101 the corresponding frequency-domain view:
(a) $H_d(e^{j\omega})$, (b) $W_h(e^{j\omega})$, and (c) $H_t(e^{j\omega}) = H_d(e^{j\omega}) \circledast W_h(e^{j\omega})$.

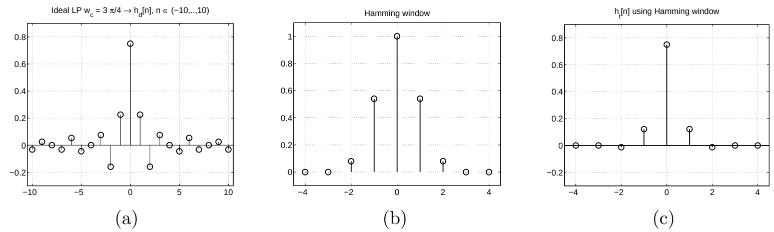


Figure 100: Problem 50(b): time domain view, (a) $h_d[n]$, (b) $w_h[n]$,(c) $h_t[n]$.

c) **Some examples of window functions:**

i) Rectangular N=11, Figure 102

ii) Rectangular N=65, Figure 103

iii) Hamming N=65, Figure 104

[2]The expression is slightly different from that given in (*Mitra 2Ed Eq. 7.75, p. 452 / 3Ed Eq. 10.31, p. 533*) but the same as in Matlab.

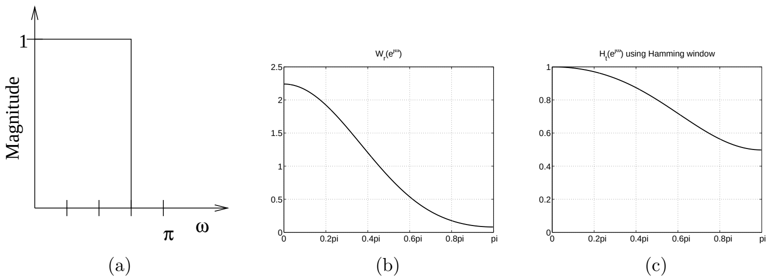


Figure 101: Problem 50(b): frequency domain $(0 \ldots \pi)$, (a) $H_d(e^{j\omega})$, (b) $W_h(e^{j\omega})$, (c) $H_t(e^{j\omega})$.

There are three figures for each item. Top left figure is the window function in time domain $w[n]$. The causal version can be obtained by shifting. Bottom left figure is the window function in frequency domain $W(e^{j\omega})$. The third figure in right is the amplitude frequency of actual filter which is obtained via window function method. The desired lowpass filter $H_d(e^{j\omega})$ is drawn in dashed line, the implemented filter $H_t(e^{j\omega}) = H_d(e^{j\omega}) \circledast W(e^{j\omega})$ is solid line. The cut-off frequency is at 100 Hz, and the sampling frequency is 1000 Hz.

Notice that

i) Rectangular N=11 gives insufficient result.

ii) Rectangular N=65 gives sharp transition band but oscillates (Gibbs phenomenon).

iii) Hamming N=65 is flat both in passband and stopband but the transition band is not as tight as in (ii).

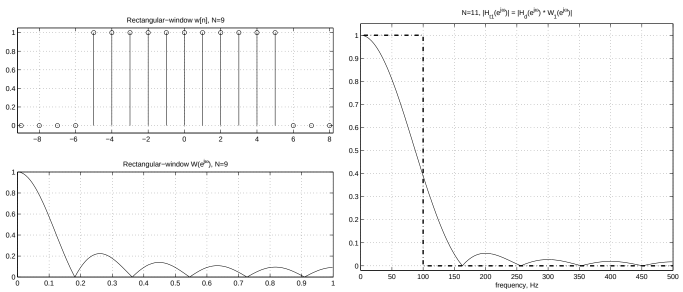


Figure 102: Rectangular window $N = 11$, see the text in Problem 50(c).

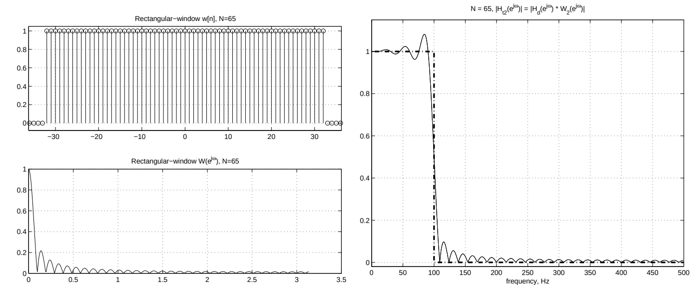


Figure 103: Rectangular window $N = 65$, see the text in Problem 50(c).

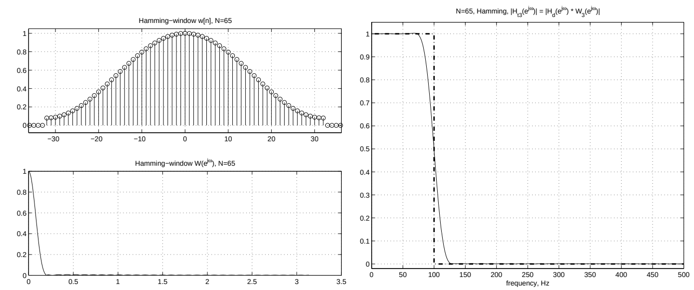


Figure 104: Hamming window $N = 65$, see the text in Problem 50(c).

51. **Problem:** The following transfer functions $H_1(z)$ and $H_2(z)$ representing two different filters meet (almost) identical amplitude response specifications

$$H_1(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

where $b_0 = 0.1022$, $b_1 = -0.1549$, $b_2 = 0.1022$, $a_1 = -1.7616$, and $a_2 = 0.8314$, and

$$H_2(z) = \sum_{k=0}^{12} h[k] z^{-k}$$

where $h[0] = h[12] = -0.0068$, $h[1] = h[11] = 0.0730$, $h[2] = h[10] = 0.0676$, $h[3] = h[9] = 0.0864$, $h[4] = h[8] = 0.1040$, $h[5] = h[7] = 0.1158$, $h[6] = 0.1201$.

For each filter,

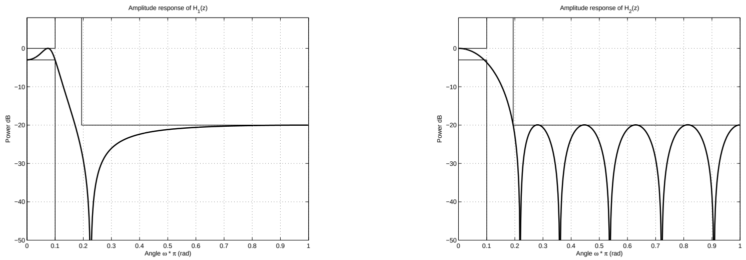
a) state if it is a FIR or IIR filter, and what is the order

b) draw a block diagram and write down the difference equation

c) determine and comment on the computational and storage requirements

d) determine first values of $h_1[n]$

Solution: The transfer functions $H_1(z)$ and $H_2(z)$ have been designed using the same amplitude specifications, see Figure 105.



$|H_1(e^{j\omega})|$ $|H_2(e^{j\omega})|$

Figure 105: Amplitude responses of $H_1(z)$ and $H_2(z)$ in Problem 51.

a) $H_1(z)$ is IIR. There is a denominator polynomial.
$H_2(z)$ is FIR. There is only the nominator polynomial.

b) $H_1(z)$ is an IIR filter. In order to show the feedback in time domain one has to use inverse z-transform:

$$\begin{aligned}
H(z) = \frac{Y(z)}{X(z)} &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \\
Y(z)\big(1 + a_1 z^{-1} + a_2 z^{-1}\big) &= X(z)\big(b_0 + b_1 z^{-1} + b_2 z^{-1}\big) \qquad | \ Z^{-1}\{.\} \\
y[n] + a_1 y[n-1] + a_2 y[n-2] &= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]
\end{aligned}$$

From the difference equation the block diagram can be drawn (Figure 106). Note that the same coefficients can be found also in the form of $H_1(z)$.

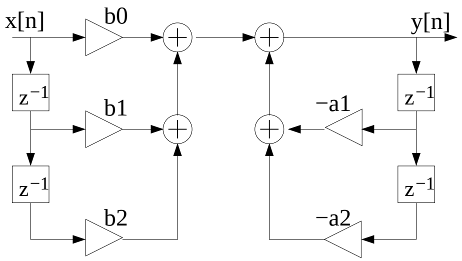


Figure 106: $H_1(z)$ as a block diagram in Problem 51.

The impulse response $h[n]$ of FIR filter $H_2(z)$ is directly seen and its length is 13 (finite impulse response). The block diagram consists only of multipliers and delays (Figure 107).

c) From examination of the two difference equations the computational and storage requirements for both filters are summarized in Table 10.
It is evident that the IIR filter is more econimical in both computational and storage requirements than the FIR filter. However, there are some tricks to improve FIR filter structure, see e.g. (*Mitra 2Ed Sec. 6.3.3, 6.3.4 / 3Ed Sec. 8.3.3, 8.3.4*)

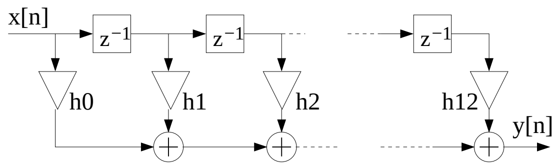


Figure 107: $H_2(z)$ as a block diagram in Problem 51.

	FIR	IIR
Number of multiplications	13	5
Number of additions	12	4
Storage locations (coefficients and data)	26	10

Table 10: Computational and storage requirements of $H_1(z)$ and $H_2(z)$.

d) A simple way to determine the impulse response is to insert an impulse $x[n] = \delta[n]$ into input and compute recursively with difference equation what comes out in $y[n]$. The registers are assumed to be zero in the initial moment. Another way to solve first values of $h_1[n]$ is to apply long division. Unfortunately, both cases are heavy by hands. Inverse z-transform can be used in order to receive exact $h[n]$. Using Matlab,

$$h_1[n] = \{\underline{0.1022},\ 0.0251,\ 0.0615,\ 0.0875,\ 0.1029,\ 0.1086,\ \ldots\}$$

52. **Problem:** Suppose that the calculation of FFT for a one second long sequence, sampled with 44100 Hz, takes 0.1 seconds. Estimate the time needed to compute (a) DFT of a one second long sequence, (b) FFT of a 3-minute sequence, (c) DFT of a 3-minute sequence. The complexities of DFT and FFT can be approximated with $\mathcal{O}(N^2)$ and $\mathcal{O}(N \log_2 N)$, respectively.

Solution: Fast Fourier Transform (FFT) is a computationally effective algorithm for calculating the Discrete Fourier Transform (DFT) of a sequence (*Mitra 2Ed Sec. 8.3.2 / 3Ed Sec. 11.3.2*).

The computational complexity of FFT is $\mathcal{O}(N \log N)$ where N is the length of the sequence. The complexity of the basic algorithm for DFT is quadratic to the input length i.e. $\mathcal{O}(N^2)$.

Here, it is supposed that the calculation of FFT for a one second long sequence, sampled with 44100 Hz, takes 0.1 seconds. Thus, the length of the sequence is $N = 1\text{ s} \times 44100\text{ Hz} = 44100$ samples and we can approximate the number of operations needed for the calculation as $N \log_2 N$ (using the base-2 logarithm). Since performing these operations takes 0.1 seconds, we get the (average) execution time for a single operation:

$$t = \frac{0.1\text{s}}{44100 \log_2(44100)} \approx 147\text{ ns}$$

a) The time needed to compute DFT of a one second long sequence is estimated as the number of operations needed times the execution time for a single operation:

$$N^2 t = 44100^2 \times 147\text{ ns} \approx 300\text{ s} \approx 5\text{ min}$$

b) A 3-minute sequence, sampled with 44100 Hz, consists of $N' = 180\text{ s} \times 44100\text{ Hz} = 7938000$ samples. Calculating FFT for N' takes approximately:

$$N' \log_2(N') t = 7938000 \log_2(7938000) \times 147\text{ ns} \approx 30\text{ s}$$

c) Calculating DFT for N' takes approximately:

$$(N')^2 t = 7938000^2 \times 147\text{ ns} \approx 9 \cdot 10^6\text{ s} \approx 100\text{ d}$$

It should be noted that these are only very crude approximations of the actual time it takes to calculate the FFT and DFT algorithms with different sizes of input sequences. The $\mathcal{O}(\cdot)$ notation omits all additive constants and constant coefficients of the complexity and concerns only the asymptotic behavior of complexity when N grows without limit. In addition, the length of N is assumed to be a power of 2 in FFT algoritms.

53. **Problem:** Express the desimal number -0.3125 as a binary number using sign bit and four bits for the fraction in the format of (a) sign-magnitude, (b) ones' complement, (c) two's complement. What would be the value after truncation, if only three bits are saved.

Solution: The binary number representation is discussed in (*Mitra 2Ed Sec. 8.4 / 3Ed Sec. 11.8*). Now, $-0.3125 = -5/16$. We can express it in fixed-point representation using a sign bit s and four bits for the fraction.

There are three different forms for negative numbers, for which all the sign bit is 0 for a positive number and 1 for a negative number.

a) Sign-magnitude format: $1_\Delta 0101$.
b-bit fraction is always $\sum_{i=1}^{b} a_{-i} 2^{-i}$. For a negative number $s = 1$:
$S = -(0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3} + 1 \cdot 2^{-4}) = -0.3125$.

b) Ones' complement: $1_\Delta 1010$.
Decimal number $S = -s(1 - 2^{-b}) + \sum_{i=1}^{b} a_{-i} 2^{-i}$. The negative number can also be achieved by complementing all bits of the corresponding positive value ($+0.3125 \triangleq 0_\Delta 0101 \rightarrow 1_\Delta 1010 \triangleq -0.3125$).
$S = -1(1 - 2^{-4}) + (1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} + 0 \cdot 2^{-4})$
$= -0.9375 + 0.625 = -0.3125$

c) Two's complement: $1_\Delta 1011$.
Decimal number $S = -s + \sum_{i=1}^{b} a_{-i} 2^{-i}$. It can also be achieved by complementing all bits and adding 1 to the least-significant bit (LSB) ($+0.3125 \triangleq 0_\Delta 0101 \rightarrow 1_\Delta 1010 + 1 = 1_\Delta 1011 \triangleq -0.3125$).
$S = -1 + (1 \cdot 2^{-1} + 0 \cdot 2^{-2} + 1 \cdot 2^{-3} + 1 \cdot 2^{-4})$
$= -1 + 0.6875 = -0.3125$

The two's complement is normally used in DSP chips.

After truncation

a) $1_\Delta 0101 \rightarrow 1_\Delta 01 \triangleq -0.25$

b) $1_\Delta 1010 \rightarrow 1_\Delta 10 \triangleq -0.25$

c) $1_\Delta 1011 \rightarrow 1_\Delta 10 \triangleq -0.5$

it can be seen that in this case truncation of (a) and (b) produced a bigger number, but (c) a smaller. The analysis of quantization (truncation) process (*Mitra 2Ed Sec. 9.1 / 3Ed Sec. 12.1*) results to quantization errors depicted in Problem 55.

54. **Problem:** In the following Figure 108, some error probability density functions of the quantization error are depicted.

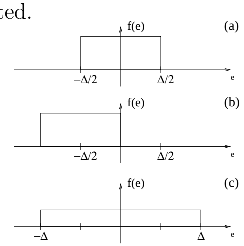


Figure 108: Problem 54: Error density functions, also at page 19.

(a) Rounding
(b) Two's complement truncation
(c) Magnitude (one's complement) truncation

is used to truncate the intermediate results. Calculate the expectation value of the quantization error m_e and the variance σ_e^2 in each case.

Solution: In this problem we are analysing different types of quantization methods. Δ here means the quantization step, $\Delta = 2^{-B}$. For example, if we are using $(B+1) = (4+1)$ bits and fixed-point numbers with two's complement representation, possible $2^{B+1} = 32$ quantized values are
$\{-1, -15/16, -14/16, \ldots, 14/16, 15/16\}$.

The area (integral) of the propability density function $f(e)$ is always one. All the distributions are uniform. Hence, $f(e)$ (height of the box) of each pdf is easily computed. We first compute $\mathrm{E}[E] = m_e$ and $\mathrm{Var}[E] = \mathrm{E}[(E - \mathrm{E}[E])^2] = \sigma_e^2$ for general uniform distribution (see Figure 109).

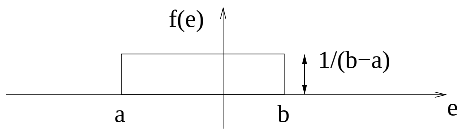


Figure 109: Computing the mean and variance of general uniform distribution in Problem 54.

$$f(e) = \begin{cases} \dfrac{1}{b-a} & a \le e \le b \\ 0 & e < a \vee e > b \end{cases}$$

$$\begin{aligned} m_e &= \int_{-\infty}^{\infty} e f(e) de = \int_a^b e \frac{1}{b-a} de = \frac{1}{b-a} \Big/_{a}^{b} \frac{1}{2} e^2 \\ &= \frac{1}{2}\frac{1}{b-a}(b^2 - a^2) = \frac{1}{2}\frac{1}{b-a}(b-a)(b+a) = \frac{1}{2}(b+a) \end{aligned}$$

$$\begin{aligned} \sigma_e^2 &= \int_{-\infty}^{\infty} (e - m_e)^2 f(e) de = \int_a^b \left[e - \frac{1}{2}(a+b)\right]^2 \frac{1}{b-a} de \\ &= \frac{1}{b-a} \Big/_{a}^{b} \frac{1}{3}\left[e - \frac{1}{2}(a+b)\right]^3 \\ &= \frac{1}{3}\frac{1}{b-a}\left\{\left[b - \frac{1}{2}(a+b)\right]^3 - \left[a - \frac{1}{2}(a+b)\right]^3\right\} \\ &= \frac{1}{3}\frac{1}{b-a}\left\{\left[\frac{1}{2}b - \frac{1}{2}a\right]^3 - \left[\frac{1}{2}a - \frac{1}{2}b\right]^3\right\} \\ &= \frac{1}{12}\frac{1}{b-a}(b-a)^3 = \frac{1}{12}(b-a)^2 \end{aligned}$$

Computation of mean and variance for each tree cases in the exercise paper, (a) rounding, (b) two's complement truncation, and (c) magnitude truncation.

a) Rounding: $a = -\frac{\Delta}{2}$, $b = \frac{\Delta}{2}$

$$m_e = \frac{1}{2}(-\frac{\Delta}{2} + \frac{\Delta}{2}) = 0$$

$$\sigma_e^2 = \frac{1}{12}\left[\frac{\Delta}{2} - \left(-\frac{\Delta}{2}\right)\right]^2 = \frac{\Delta^2}{12}$$

b) Two's complement truncation: $a = -\Delta$, $b = 0$

$$m_e = \frac{1}{2}(-\Delta + 0) = -\frac{\Delta}{2}$$

$$\sigma_e^2 = \frac{1}{12}[0 - (-\Delta)]^2 = \frac{\Delta^2}{12}$$

c) Magnitude truncation: $a = -\Delta$, $b = \Delta$

$$m_e = \frac{1}{2}(-\Delta + \Delta) = 0$$

$$\sigma_e^2 = \frac{1}{12}[\Delta - (-\Delta)]^2 = \frac{\Delta^2}{3}$$

55. **Problem:** In this problem we study the roundoff noise in direct form FIR filters. Consider an FIR filter of length N having the transfer function

$$H(z) = \sum_{k=0}^{N-1} h[k]z^{-k}.$$

Sketch the direct form realization of the transfer function.

a) Derive a formula for the roundoff noise variance when quantization is done before summations.

b) Repeat (a) for the case where quantization is done after summations, i.e. a double precision accumulator is used.

Solution: Direct form realization of the filter. Quantization blocks are marked by Q in Figure 110.

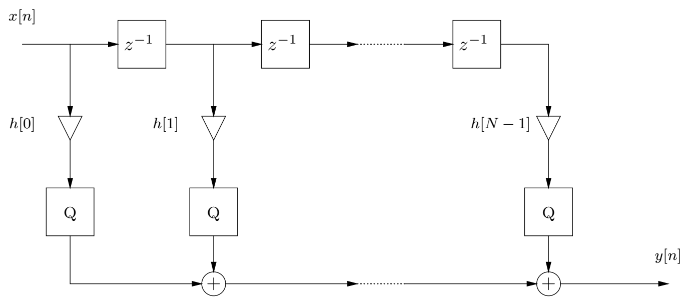


Figure 110: Filter with finite wordlength in Problem 55.

a) The roundoff noise model ($e_i[n]$:s are error sources), when quantization is done before summations, is depicted in Figure 111.

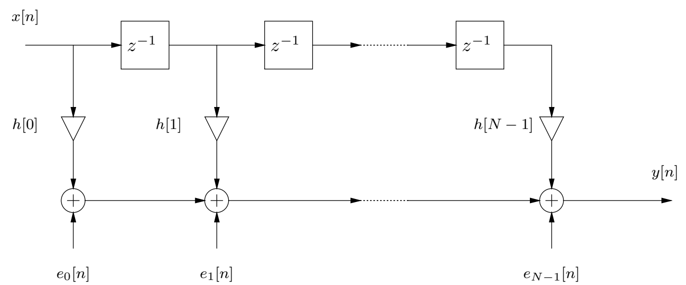


Figure 111: Roundoff noise model with N quantization points in Problem 55.

It is assumed that the quantization is done using rounding. $B+1$ bits are used in the coefficient quantization ($\Delta = 2^{-B}$):

$$\Rightarrow \sigma_e^2 = \frac{2^{-2B}}{12}, \quad m_e = 0 \text{ for all } e_i[n], \quad i = 0, \ldots, N-1.$$

Transfer functions from noise sources to the output are equal to unity. Total output noise is thus

$$e[n] = \sum_{i=0}^{N-1} e_i[n] \ .$$

The variance of the noise is

$$\begin{aligned}
\sigma_{e,tot}^2 &= E\left[e^2[n]\right] - \underbrace{E[e[n]]}_{=0 \text{ (rounding)}}{}^2 \\
&= E\left[\left(\sum_{i=0}^{N-1} e_i[n]\right)^2\right] \qquad\qquad [E[e_i[n]e_j[n]] = 0, i \neq j] \\
&= \sum_{i=0}^{N-1} E\left[e_i^2[n]\right] = \sum_{i=0}^{N-1} \sigma_e^2 = N\sigma_e^2 = N\frac{2^{-2B}}{12}
\end{aligned}$$

b) The model, when quantization is done after summations, is drawn in Figure 112. Now there is only one quantization point, i.e., there is only one noise source, $e[n]$.

$$\Rightarrow \sigma_{e,tot}^2 = \sigma_e^2 = \frac{2^{-2B}}{12}.$$

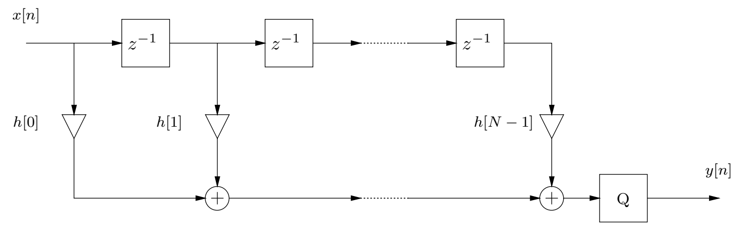


Figure 112: Filter with only one quantization point in Problem 55.

56. **Problem:** Consider a lowpass DSP system with a second-order noise reduction system in Figure 113(a).

   a) What is the transfer function of the system if infinite wordlength is used?
   b) Derive an expression for the transform of the quantized output, $Y(z)$, in terms of the input transform, $X(z)$, and the quantization error, $E(z)$, and hence show that the error feedback network has no adverse effect on the input signal.
   c) Deduce the expression for the error feedback function.
   d) What values k_1 and k_2 should have in order to work as an error-shaping system?

Solution: The quantization errors produced in digital systems may be compensated by error-shaping filters. First-order and second-order feedback structures are introduced in (*Mitra 2Ed Sec. 9.10.1, 9.10.2 / 3Ed Sec. 12.10.1, 12.10.2*). The error components are extracted from the system and processed e.g. using simple digital filters. This way the noise at the output of the system can be reduced.

Consider first the block diagram shown in Figure 113(a) and its round-off noise model in Figure 113(b).

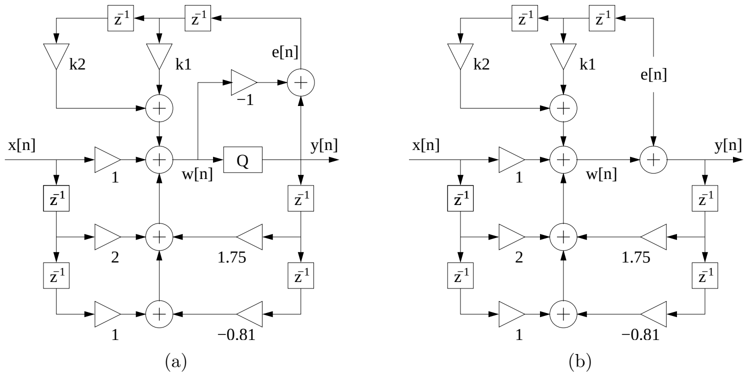


Figure 113: (a) Second-order direct form I system with second-order noise reduction, (b) and its noise model in Problem 56.

a) If infinite precision is used, the quantization is not needed and $e[n] \equiv 0$ (see Figure 113(b) with $e[n] = 0$). In that case, the system function is

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 1.75z^{-1} + 0.81z^{-2}}$$

Computing zeros and poles we get a pole-zero diagram from which it can be derived that the filter is lowpass (Figure 114).

b) From Figure 113(a) it can be obtained the following difference equations:

$$\begin{aligned} e[n] &= y[n] - w[n] \\ w[n] &= (x[n] + 2x[n-1] + x[n-2]) \\ &\quad + (1.75y[n-1] - 0.81y[n-2]) \\ &\quad + (k_1 e[n-1] + k_2 e[n-2]) \end{aligned}$$

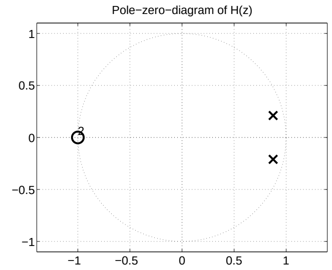


Figure 114: The pole-zero plot of $H(z) = (1+2z^{-1}+z^{-2})/(1-1.75z^{-1}+0.81z^{-2})$ in Problem 56.

After z-transform,

$$\begin{aligned} Y(z) &= \left[\frac{1 + 2z^{-1} + z^{-2}}{1 - 1.75z^{-1} + 0.81z^{-2}}\right] X(z) + \left[\frac{1 + k_1 z^{-1} + k_2 z^{-2}}{1 - 1.75z^{-1} + 0.81z^{-2}}\right] E(z) \\ &= H(z)X(z) + H_e(z)E(z) \end{aligned}$$

It can be observed that the noise transfer function $H_e(z)$ modifies only the quantization error.

c) The noise transfer function is

$$H_e(z) = \frac{1 + k_1 z^{-1} + k_2 z^{-2}}{1 - 1.75z^{-1} + 0.81z^{-2}} = H_{eu}(z)\, H_{es}(z)$$

Notice that without error-shaping feedback structure, i.e., $k_1 = 0$ and $k_2 = 0$, the noise transfer function is (u = unshaped)

$$H_{eu}(z) = \frac{1}{1 - 1.75z^{-1} + 0.81z^{-2}}$$

So, the error-feedback circuit is actually shaping the error spectrum by (s = shaping)

$$H_{es}(z) = 1 + k_1 z^{-1} + k_2 z^{-2}$$

d) Without error-shaping the quantized output spectrum is

$$Y_u(z) = H(z)X(z) + H_{eu}(z)E(z)$$

Error-shaping filter $H_{es}(z)$ should efficiently discard the effects of the poles of $H_{eu}(z)$. Error-feedback coefficients are chosen to be simple integers or fractions ($k_i = 0, \pm 0.5, \pm 1, \pm 2$), so that the multiplication can be performed using a binary shift operation and it will not introduce an additional quantization error. Choosing $k_1 = -2$, $k_2 = 1$, $H_{es}(z) = 1 - 2z^{-1} + z^{-2}$ is a highpass filter with two zeros at $z = 1$.

The error shaping structure lowers the noise in the passband by pushing it into the stopband of the filter (*Mitra 2Ed Fig. 9.45 / 3Ed Fig. 12.46*).

57. **Problem:** Consider a cosine sequence $x[n] = \cos(2\pi(f/f_s)n)$ where $f = 10$ Hz and $f_s = 100$ Hz as depicted in the top left in Figure 115. While it is a pure cosine, its spectrum is a peak at the frequence $f = 10$ Hz (top middle) or at $\omega = 2\pi f/f_s = 0.2\pi$ (top right).

   a) Sketch the output sequence $x_u[n]$ and its spectra using up-sampler with up-sampling factor $L = 2$.

   b) Sketch the output sequence $x_d[n]$ and its spectra using down-sampler with factor $M = 2$.

Solution: Sometimes it is necessary or useful to change the sampling frequency f_s. Consider music formats DAT (48 kHz) and CD (44.1 kHz).

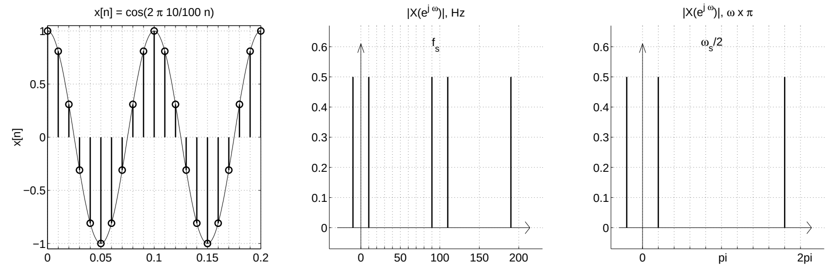


Figure 115: Problem 57(a). The original sequence of a cosine of $f = 10$ Hz and its spectrum. The angular frequency $\omega = 2\pi(f/f_s) = 2\pi(10/100) = 0.2\pi$.

a) Up-sampling with factor $L = 2$. In the time domain there will be $L-1$ zeros between the original samples, see Figure 116(a).

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \dots \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x[n/2], & n = 0, \pm 2, \pm 4, \dots \\ 0, & \text{otherwise} \end{cases}$$

In the frequency domain the sampling frequency is multiplied by L, hence, the new sampling frequency is 200 Hz. $L-1$ images from the original spectrum are emerged equivalently between 0 and $f_{s,new}$.

$$X_u(e^{j\omega}) = X(e^{j\omega L}) = X(e^{j2\omega})$$

Each cosine is a peak pair ($\pm f$) in the spectrum. The original peaks are at $f = 10$ and $f = 200 - 10 = 190$ Hz, and after up-sampling new images at $f = 90$ and $f = 110$ Hz, as shown in Figure 116(b). The same with angular frequencies is shown in Figure 116(c).

Notice that if you ideally convert the sequence $x_u[n]$ into continuous-time $x_u(t)$ you will find also a high frequency component, an image component. Normally images are filtered out using a lowpass filter (see anti-imaging and anti-aliasing filters). See Figure 117.

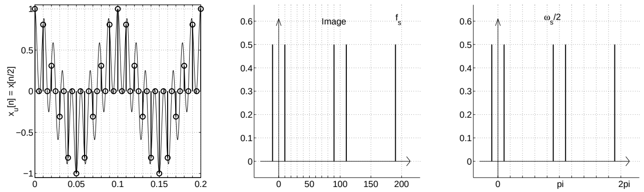


Figure 116: Problem 57(a). Up-sampled signal $x_u[n]$, factor $L = 2$. The sampling frequency is increased to 200 Hz, and there is an image spectrum.

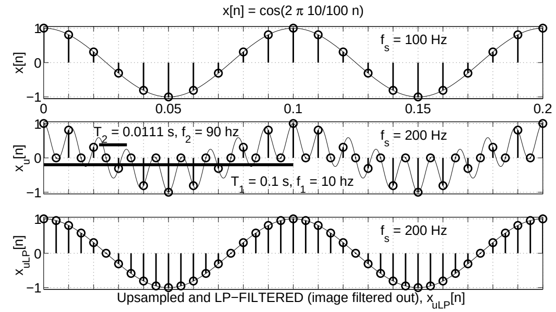


Figure 117: A closer look at up-sampling. Top, original sequence. Middle $L = 2$, $L - 1 = 1$ zeros added between the original samples. Bottom, using (ideal) LP-filter to remove the image, i.e., 90 Hz component. The continuous curve is plotted only for better visual view. See the text in Problem 57(a).

b) Down-sampling with factor $M = 2$ means taking only every second sample.

$$x_d[n] = x[nM] = x[2n]$$

A possible effect is losing information. However, in this case, this does not occur because $f = 10$ Hz $< f_{s,new}/2 = 25$ Hz. See Figure 118(a).

In the frequency domain the sampling frequency is decreased to 50 Hz. See Figures 118(b)-(c).

$$X_d(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j(\omega - 2\pi k)/M})$$

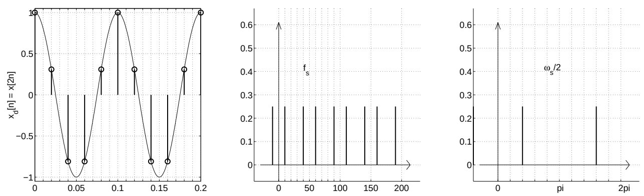


Figure 118: Problem 57(b). Down-sampled signal $x_d[n]$, factor $M = 2$. The sampling frequency is decreased to 50 Hz.

58. **Problem:** Express the output $y[n]$ of the system shown in Figure 119 as a function of the input $x[n]$.

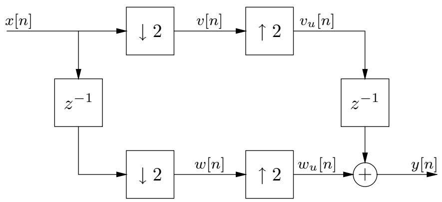


Figure 119: Multirate system of Problem 58.

Solution: Consider an input signal $x[n]$ with the corresponding z-transform $X(z)$. After factor-of-L up-sampling, the z-transform of the signal $x_u[n]$ is

$$X_u(z) = X(z^L)$$

and after factor-of-M down-sampling, the z-transform of the signal $x_d[n]$ is

$$X_d(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W_M^{-k})$$

where $W_M = e^{-j2\pi/M}$. See (*Mitra 2Ed Sec. 10.1.2 / 3Ed Sec. 13.1.2*) for the derivation of these equations.

Using these equations, let us derive the z-transforms of the intermediate signals $v[n]$, $v_u[n]$, $w[n]$, and $w_u[n]$ and finally the z-transform of the output $y[n]$. Let us denote the delayed version of the input as $X'(z) = z^{-1}X(z)$. Furthermore, note that $W_2^{-1} = e^{j2\pi/2} = -1$.

$$\begin{aligned}
V(z) &= \frac{1}{2}\sum_{k=0}^{1} X(z^{1/2}W_2^{-k}) = \frac{1}{2}X(z^{1/2}) + \frac{1}{2}X(-z^{1/2}) \\
W(z) &= \frac{1}{2}\sum_{k=0}^{1} X'(z^{1/2}W_2^{-k}) = \frac{1}{2}z^{-1/2}X(z^{1/2}) - \frac{1}{2}z^{-1/2}X(-z^{1/2}) \\
V_u(z) &= V(z^2) = \frac{1}{2}X(z) + \frac{1}{2}X(-z) \\
W_u(z) &= W(z^2) = \frac{1}{2}z^{-1}X(z) - \frac{1}{2}z^{-1}X(-z) \\
Y(z) &= z^{-1}V_u(z) + W_u(z) = z^{-1}X(z)
\end{aligned}$$

or $y[n] = x[n-1]$ in time-domain (derive the same in time-domain!).

59. **Problem:** Show that the factor-of-L up-sampler $x_u[n]$ and the factor-of-M down-sampler $x_d[n]$ defined as in Problem 57 are linear systems.

Solution: First, consider the up-sampler. Let $x_1[n]$ and $x_2[n]$ be two arbitrary inputs with $y_1[n]$ and $y_2[n]$ as the corresponding outputs. Now,

$$\begin{aligned}
y_1[n] &= \begin{cases} x_1[n/L] & : \ n = 0, \pm L, \pm 2L, \ldots \\ 0 & : \ \text{otherwise} \end{cases} \\
y_2[n] &= \begin{cases} x_2[n/L] & : \ n = 0, \pm L, \pm 2L, \ldots \\ 0 & : \ \text{otherwise} \end{cases}
\end{aligned}$$

Let us now apply the input $x_3[n] = \alpha x_1[n] + \beta x_2[n]$ with the corresponding output $y_3[n]$ as

$$\begin{aligned}
y_3[n] &= \begin{cases} \alpha x_1[n/L] + \beta x_2[n/L] & : \ n = 0, \pm L, \pm 2L, \ldots \\ 0 & : \ \text{otherwise} \end{cases} \\
&= \begin{cases} \alpha x_1[n/L] \\ 0 \end{cases} + \begin{cases} \beta x_2[n/L] & : \ n = 0, \pm L, \pm 2L, \ldots \\ 0 & : \ \text{otherwise} \end{cases} \\
&= \alpha y_1[n] + \beta y_2[n]
\end{aligned}$$

Thus, the up-sampler is a linear system.

Now, consider the down-sampler with the inputs $x_1[n]$ and $x_2[n]$ and the corresponding outputs $y_1[n]$ and $y_2[n]$. Now, $y_1[n] = x_1[nM]$ and $y_2[n] = x_2[nM]$. By applying the input $x_3[n] = \alpha x_1[n] + \beta x_2[n]$ we get the corresponding output $y_3[n] = x_3[nM] = \alpha x_1[nM] + \beta x_2[nM]$. Hence, the down-sampler is also a linear system.

It should also be noted, that both the up-sampler and the down-sampler are time-varying, i.e. not LTI systems.

60. **Problem:** Consider the multirate system shown in Figure 120 where $H_0(z)$, $H_1(z)$, and $H_2(z)$ are ideal lowpass, bandpass, and highpass filters. Sketch the Fourier transforms of the outputs $y_0[n]$, $y_1[n]$, and $y_2[n]$ if the Fourier transform of the input is as shown in Figure 121(a).

    Solution: First, let us denote the down-sampled signal as $x_d[n]$ and the again up-sampled signal as $x_u[n]$, shown in Figure 120.

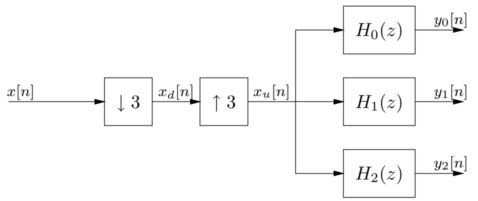


Figure 120: The multirate system in Problem 60.

The corresponding Fourier transforms (spectra) $X_d(z)$ and $X_u(z)$ are as follows (notice the reduced amplitude) in Figure 121.

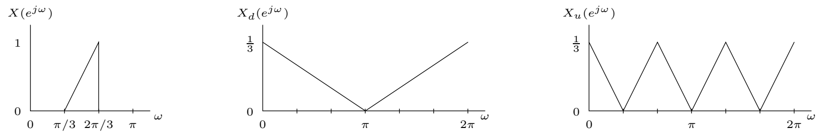


Figure 121: Original, upsampled and downsampled spectrum in Problem 60.

Now, the Fourier transforms of the outputs $Y_0(z)$, $Y_1(z)$, and $Y_2(z)$, are obtained by (ideally) filtering $X_u(z)$. The output spectra are in Figure 122.

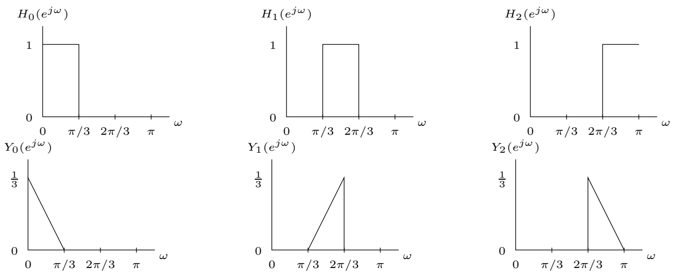


Figure 122: Bandpass filters in top row, and corresponding Output spectra in bottom row in Problem 60.