Datasta Tietoon, Autumn 2007

SOLUTIONS TO EXERCISES 1

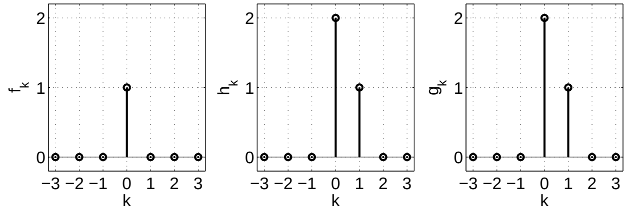
H1 / Problem 1.
a)

$$g_k = \sum_{m=-\infty}^{\infty} f_m h_{k-m}$$

with

$$f_0 = 1,\ f_m = 0 \text{ otherwise;} \qquad (1)$$
$$h_0 = 2,\ h_1 = 1,\ h_n = 0 \text{ otherwise} \qquad (2)$$

Thus $g_k = f_0 h_{k-0} = h_k$, which is $g_0 = 2$, $g_1 = 1$, and $g_k = 0$ elsewhere.



b)

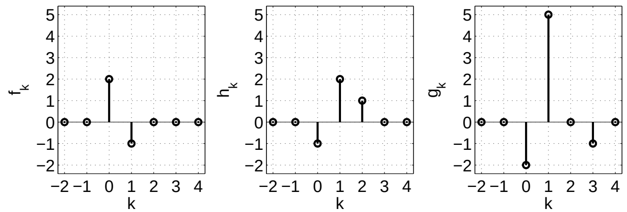
$$f_0 = 2,\ f_1 = -1,\ f_m = 0 \text{ otherwise;} \qquad (3)$$
$$h_0 = -1,\ h_1 = 2,\ h_2 = 1,\ h_n = 0 \text{ otherwise.} \qquad (4)$$

Thus

$$g_k = f_0 h_{k-0} + f_1 h_{k-1} = 2h_k - h_{k-1}$$

and we get

$$g_3 = 2h_3 - h_2 = -1 \qquad (5)$$
$$g_2 = 2h_2 - h_1 = 2 - 2 = 0 \qquad (6)$$
$$g_1 = 2h_1 - h_0 = 4 + 1 = 5 \qquad (7)$$
$$g_0 = 2h_0 - h_{-1} = -2 \qquad (8)$$
$$g_k = 0 \quad \text{otherwise} \qquad (9)$$



H1 / Problem 2.
a) Substitute $F(\omega)$ into the integral:

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\sum_{m=-\infty}^{\infty} f_m e^{-j\omega m}]e^{j\omega n}d\omega = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} f_m \int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega$$

with $j = \sqrt{-1}$ the imaginary unit (often also denoted i).
For the integral we get (note that $n, m \in \mathbb{Z}$)

$$\int_{-\pi}^{\pi} e^{j\omega(n-m)}d\omega = \begin{cases} 2\pi & \text{if } n = m, \\ \Big/_{-\pi}^{\pi} \frac{1}{j(n-m)} e^{j\omega(n-m)} = \frac{1}{j(n-m)}\left(e^{j\pi(n-m)} - e^{-j\pi(n-m)}\right) & \text{if } n \neq m \end{cases}$$

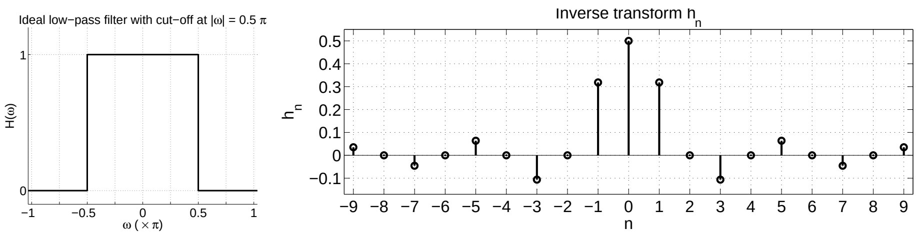
But $e^{j\pi(n-m)} = e^{-j\pi(n-m)}$ because the period of the function is 2π, thus the integral is 2π if $n = m$ and zero otherwise. Substituting this into the full expression gives $I = f_n$ which was to be shown.
b)

$$h_n = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega = \frac{1}{2\pi}\Big/_{-\omega_0}^{\omega_0}\frac{1}{jn}e^{j\omega n} \qquad (10)$$
$$= \frac{1}{2\pi jn}(e^{j\omega_0 n} - e^{-j\omega_0 n}) \qquad (11)$$
$$= \frac{1}{2\pi jn}[\cos(\omega_0 n) + j\sin(\omega_0 n) - \cos(\omega_0 n) + j\sin(\omega_0 n)] \qquad (12)$$
$$= \frac{1}{\pi n}\sin(\omega_0 n). \qquad (13)$$

Using the cut-off frequency $\omega = \pi/2$ we get

$$h_n = \frac{1}{\pi n}\sin(\frac{\pi n}{2})$$

Some values: $h_0 = 0.5$, $h_1 = 1/\pi$, $h_2 = 0$. Note that $(1/x)\sin(x/2) = (1/2)(2/x)\sin(x/2) = (1/2) - (x^2/48) + \ldots$ (Taylor). Thus at zero the value is 0.5. Note also that the sequence h_n is infinitely long.



H1 / Problem 3.

Now the number of bins is at most 100000, because the average number of substrings in a bin must be at least 10. The number of different substrings of length n is 4^n. We get

$$4^n \leq 100000$$

giving $n \leq 8$.

H1 / Problem 4.

The volume of the unit hypercube is 1 and the volume of the set of inner points is $(1-2\epsilon)^d$. For any ϵ, this tends to 0 as $n \to \infty$.

H1 / Problem 5.

Now the small hypercubes are similar, hence all have the same volume which must be $\frac{1}{n}$ times the volume of the large unit hypercube. (This is only possible for certain values of (n, d); for $d = 2$, n must be 4, 9, 16, ...; for $d = 3$, n must be 8, 27, 64 ... etc.)

Also, we assume here a special distance which is not Euclidean distance but $D(\mathbf{x}_1, \mathbf{x}_2) = \max_i |x_{i1} - x_{i2}|$, that is, the largest distance along the coordinate axes.

Then it is easy to see that the distance of the centres of the small hypercubes is equal to the length of their side s. Because the volume is $s^d = \frac{1}{n}$, we have $s = \frac{1}{n}^{\frac{1}{d}}$.

The case of $d = 2, n = 4$ is shown below.