Helsinki University of Technology
Lab of Computer and Information Science

Tik-61.184 Special Course in Information Technology
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Exercises, Session 2

1. Consider the following distribution

$$g(x) = \frac{b}{4}\{exp(-b|x-a|) + exp(-b|x+a|)\}$$

   (a) Using the general expression for the n^{th} order moment of a distribution $p(x)$, which has infinite support, $m_n = \int_{-\infty}^{\infty} p(x)x^n dx$ show that the kurtosis of $g(x)$ is defined by the expression

$$\frac{12 - 2a^4b^4}{4 + 4a^2b^2 + a^4b^4}$$

   (b) Study the values of kurtosis for this distribution when the parameters a and b take on a range of values.

      i. When is the value of the kurosis negative?

      ii. When is the value of the kurtosis positive?

      iii. When is the value of the kurtosis zero ?

   (c) Is this distribution Sub or Super-Gaussian? discuss.

2. The general linear model is defined by $\mathbf{x} = \mathbf{A}\mathbf{s} + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon}$ is a random element defining the noise inherent in the model and $\mathbf{A}$ may not be square. The standard ICA model $\mathbf{x} = \mathbf{A}\mathbf{s}$ where $\mathbf{A}$ is square can be viewed as a version of this model where noise is unaccounted for.

   The ICA model where $\mathbf{A}$ is square $\mathbf{x} = \mathbf{A}\mathbf{s}$ admits a log-likelihood of $log(p(\mathbf{x}|\mathbf{A})) = log(p(\mathbf{s})) - log(|det(\mathbf{A})|)$

   The Laplace approximation computed at the posterior mode $\hat{\mathbf{s}}$ for the log-likelihood of the general linear model where $\boldsymbol{\epsilon}$ is distributed as a zero-mean Gaussian with covariance $\mathbf{R}$ yields the following expression.

$$p(\mathbf{x}|\mathbf{A}) \approx \frac{(2\pi)^{\frac{L}{2}}}{\sqrt{-\nabla\nabla log(f(\hat{\mathbf{s}}))}} f(\hat{\mathbf{s}})$$

   Where $f(\hat{\mathbf{s}}) = p(\hat{\mathbf{s}})p(\mathbf{x}|\hat{\mathbf{s}}, \mathbf{A})$. Note that $p(\mathbf{x}|\hat{\mathbf{s}}, \mathbf{A})$ is distributed as $\boldsymbol{\epsilon}$ the noise in the model.

   (a) Show that $-\nabla\nabla log(f(\hat{\mathbf{s}})) = \mathbf{A}^T\mathbf{R}^{-1}\mathbf{A} - \nabla\nabla log(p(\hat{\mathbf{s}}))$

   (b) From the above derive an approximate expression for the log-likehood $log(p(\mathbf{x}|\mathbf{A}))$ of the linear model.

   (c) In the vanishing noise limit give the value of $-\nabla\nabla log(f(\hat{\mathbf{s}}))$.

   (d) For the case where $\mathbf{A}$ is now defined as an $L \times L$ square matrix and using the answers to (b) and (c) show that the log-likehood for the simplified ICA model is recovered from the Laplace approximation.