Primal-Dual Algorithm

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Introduction

- general algorithm for solving LPs
- generates specialized algorithms
  - shortest paths, max flow, min-cost flow

Algorithm (preview)

```
procedure primal-dual
   infeasible := 'no', opt := 'no'
   let π be feasible in D
   while infeasible = 'no' and opt = 'no' do
      set J = {j : π^T A_j = c_j}
      solve RP by the simplex algorithm
      if ξ_opt = 0 then
         opt := 'yes'
      else
         if π̄^T A_j ≤ 0 for all j ∉ J then
            infeasible := 'yes'
         else
            π := π + θ_1 π̄
         end if
      end if
   end while
```

Primal and Dual

$$\begin{aligned} P: \min z &= c^T x \\ Ax &= b \geq 0 \\ x &\geq 0 \end{aligned}$$

$$\begin{aligned} D: \max w &= \pi^T b \\ \pi^T A &\leq c^T \\ \pi^T &\gtrless 0 \end{aligned}$$

Complementary Slackness Conditions

$$\pi_i(a_i^T x - b_i) = 0 \quad \text{for all } i \tag{1}$$
$$(c_j - \pi^T A_j)x_j = 0 \quad \text{for all } j \tag{2}$$

- (1) is satisfied since P is in standard form
- what about (2) ?

Finding x

- Given feasible π
- Find x such that $x_j = 0$ whenever $c_j - \pi^T A_j > 0$
- let's consider only rows in A that satisfy $c_j - \pi^T A_j = 0$
- index set $J = \{j : c_j - \pi^T A_j = 0\}$
- x should satisfy the following

$$
\begin{aligned}
\sum_{j \in J} a_{ij} x_j &= b_i \quad & i = 1, \ldots, m \\
x_j &\geq 0 \quad & j \in J \\
x_j &= 0 \quad & j \notin J
\end{aligned}
$$

Restricted Primal (RP)

$$\min \xi = \sum_{i=1}^{m} x_i^a$$

$$\sum_{j \in J} a_{ij} x_j + x_i^a = b_i$$

$$x_j \geq 0 \quad j \in J$$

$$x_j = 0 \quad j \notin J$$

$$x_i^a \geq 0$$

- can be solved using ordinary simplex algorithm
- optimal solution reached if $\xi_{opt} = 0$
- if $\xi_{opt} > 0$ we should look at the dual of the restricted primal (DRP)

Dual of the Restricted Primal (DRP)

$$
\begin{aligned}
DRP: \max w = \pi^T b \\
\pi^T A_j \leq 0 \qquad & j \in J \\
\pi_i \leq 1 \quad & i = 1, \ldots, m \\
\pi_i \gtrless 0
\end{aligned}
$$

- let us solve this and let the optimal solution be $\bar{\pi}$
- corrected $\pi^* = \pi + \theta\bar{\pi}$
- new cost of D is ${\pi^*}^T b = \pi^T + \theta\bar{\pi}^T b$

Choosing θ

- when choosing θ
  - cost of D should increase
  - feasibility should be maintained
- the optimal costs of RP and DRP are equal at mutual optimality since they are a primal-dual pair
  $\bar{\pi}^T b = \xi_{opt} > 0$
- cost of D was $\pi^{*T} b = \pi^T + \theta\bar{\pi}^T b$
- $\theta > 0$ to increase the cost of D

- feasibility must be maintained
- in D we have ${\pi^*}^T A_j = \pi^T A_j + \theta \bar{\pi}^T A_j \leq c_j$
- if $\bar{\pi}^T A_j \leq 0$ for every j
  $\Rightarrow$ θ can be chosen arbitarly large
  $\Rightarrow$ D is unbounded
  $\Rightarrow$ P is infeasible
- For all $j \in J$, $\bar{\pi}^T A_j \leq 0$ since $\bar{\pi}$ is optimal
- feasibility is maintained when $\bar{\pi}^T A_j > 0$ for some $j \notin J$
- $\theta_1 = \min_{\{j \notin J | \bar{\pi}^T A_j > 0\}} \left[\frac{c_j - \pi^T A_j}{\bar{\pi}^T A_j} \right]$

Algorithm

procedure primal-dual

    infeasible := 'no', opt := 'no'
    let π be feasible in D
    while infeasible = 'no' and opt = 'no' **do**
        set $J = \{j : \pi^T A_j = c_j\}$
        solve RP by the simplex algorithm
        if $\xi_{opt} = 0$ **then**
            opt := 'yes'
        else
            if $\bar{\pi}^T A_j \leq 0$ for all $j \notin J$ **then**
                infeasible := 'yes'
            else
                $\pi := \pi + \theta_1 \bar{\pi}$
            end if
        end if
    end while

Demonstration, Shortest Path Problem

$$\min c^T f$$

$$Af = \begin{bmatrix} +1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \leftarrow \textbf{Row s}$$

$$f \geq 0$$

$$
\begin{aligned}
\max \pi_s & \\
\pi_i - \pi_j \leq c_{ij} & \quad \textit{for all arcs } (i,j) \in E \\
\pi_i \gtrless 0 & \quad \textit{for all } i \\
\pi_t = 0 &
\end{aligned}
$$

$$\min \xi = \sum_{i=1}^{m-1} x_i^a$$

$$Af + x^a = \begin{bmatrix} +1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \leftarrow \textbf{Row s}$$

$$f_j \geq 0 \quad \text{for all } j$$

$$f_j = 0 \quad j \in J$$

$$x_i^a \geq 0$$

$$
\begin{aligned}
\max \pi_s & \\
\pi_i - \pi_j \leq 0 & \quad \textit{for all arcs } (i,j) \in J \\
\pi_i \leq 1 & \quad \textit{for all } i \\
\pi_i \gtreqless 0 &
\end{aligned}
$$