Geometric and algebraic interpretation

T-61.152
Informaatiotekniikan seminaari

Eero Salminen
54750N

LP assumptions

- Given LP is in standard form:
  $min\ \mathbf{c}'\mathbf{x}$
  $\mathbf{A}\mathbf{x} = \mathbf{b}$ ($\mathbf{A}$ is an $m * n$ matrix and $m < n$)
  $\mathbf{x} \geq 0$

- $\mathbf{A}$ has m linearly independent columns $\mathbf{A}_j$
  (A has rank m)

Basic solution

The *basis* of **A** is a linearly independent collection

$\beta = \{\boldsymbol{A}_{j_1}, \ldots, \boldsymbol{A}_{j_m}\} \quad \Leftrightarrow \quad \boldsymbol{B} = [\boldsymbol{A}_{j_1} \ldots \boldsymbol{A}_{j_m}] = [\boldsymbol{A}_j]$

The *basic solution* ***x*** is

$x_p = [\boldsymbol{B}^{-1}\boldsymbol{b}]_p$ for $\boldsymbol{A}_p \in \beta$

$x_q = 0$ for $\boldsymbol{A}_q \notin \beta$

Basic feasible solution

If a basic solution $\mathbf{x} \geq 0$ ($\mathbf{x} \in F$), it's a *basic feasible solution (bfs)*.

Some properties of bfs:

- There exists a $\mathbf{c}$ such that a bfs $\mathbf{x}$ is the unique optimal solution of *min* $\mathbf{c}'\mathbf{x}$ ($\mathbf{A}\mathbf{x}=\mathbf{b}$, $\mathbf{x} \geq 0$)

- When *F*, the feasible points, is not empty and $\mathbf{A}$ is of rank m, as least one bfs exists

Subspace

A (linear) subspace S of R^d is

$$S = \{\mathbf{x} \in R^d: a_{j1}x_1 + \ldots + a_{jd}x_d = 0, \;\; j = 1,\ldots,m\}$$

$$Dim(S) = d - rank([a_{ji}])$$

An affine subspace A of R^d is

$$A = \{\mathbf{x} \in R^d: a_{j1}x_1 + \ldots + a_{jd}x_d = b_j, \;\; j = 1,\ldots,m\}$$

Hyperplane

A *hyperplane* is an affine subspace of R^d of dimension d-1, the set of points in

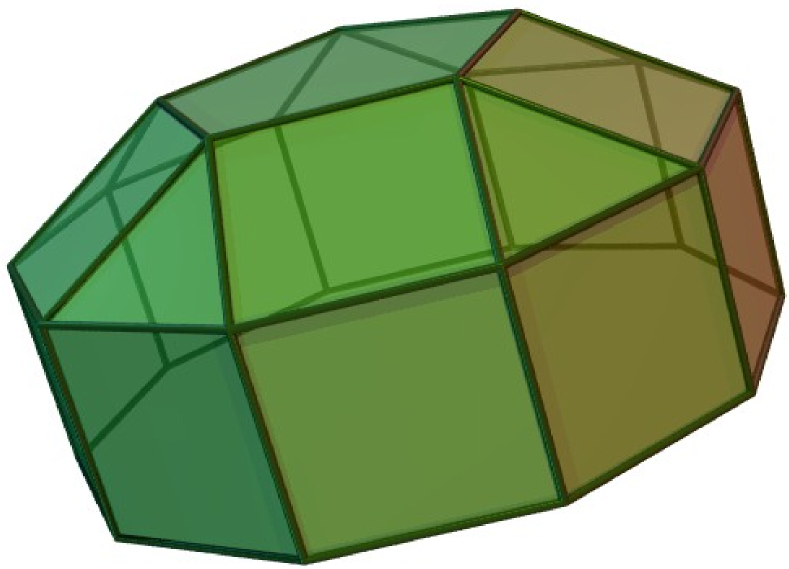
$$a_1x_1 + a_2x_2 + \ldots a_dx_d = b$$

A hyperplane defines 2 *halfspaces*

$$a_1x_1+\ldots+a_dx_d \geq b \text{ and } \leq b$$

Convex polytope

A (convex) *polytope* is a bounded intersection of finite number of halfspaces.



A face f of polytope P supported by the hyperplane H is

$$f = P \cap H$$

A facet = a face of dimension d-1
An edge = a face of dimension 1
A vertex = a face of dimension 0

Geometric views of a polytope

A convex polytope can be viewed in several different ways. The geometrical views are a bit easier to imagine:

- P is the convex hull of a finite set of points, as a polytope is the convex hull of its vertices.

- P is the intersection of *k* halfspaces

$$a_{k1}x_{k1}+\ldots+a_{kd}x_{kd} \leq b_k$$

as long as the intersection is bounded.

Slack variables

Feasible region F of a LP is $\boldsymbol{Ax=b}$, $\boldsymbol{x} \geq 0$. This can also be expressed as

$$x_i = b_i - \sum_{j=1}^{n-m} a_{ij} x_j, \quad i = n-m+1, \ldots, n$$

$$x_j \geq 0, \quad j = 1, \ldots, n-m$$

The variables x_i are also known as the *slack variables*.

Algebraic view of a polytope

By removing the slack variables we get the inequalities

$$b_i - \sum_{j=1}^{n-m} a_{ij} x_j \geq 0, \quad i = n-m+1, \ldots, n$$

$$x_j \geq 0, \quad j = 1, \ldots, n-m$$

These also define the intersection of n halfspaces, hence define a polytope P in R^{n-m}.

Polytope and the feasible set

Let P be a polytope defined by the *n* halfspaces

$$h_{i,1}x_1 + \ldots + h_{i,n-m}x_{n-m} + g_i < 0 \quad i=1,\ldots,n$$

Any point $\boldsymbol{x}_p=(x_1,\ldots,x_{n-m}) \in P$ can be transformed to $\boldsymbol{x}_f=(x_1,\ldots,x_n) \in F$ by defining:

$$x_i = -g_i - \sum_{j=1}^{n-m} h_{i,j} x_j \quad i=n-m+1,\ldots,n$$

Also any $\boldsymbol{x}_f$ can be transformed to $\boldsymbol{x}_p$ by truncating the last *m* coordinates.

Vertices of a polytope

Let P be a polytope, F the feasible set of the corresponding LP and $\boldsymbol{x}_p=(x_1,\ldots,x_{m-n}) \in P$.

Then the following are equivalent:

- Point $\boldsymbol{x}_p$ is a vertex of P
- If $\boldsymbol{x}_p=\alpha\boldsymbol{x}_{p'}+(1-\alpha)\boldsymbol{x}_{p''}$ with $\boldsymbol{x}_{p'},\boldsymbol{x}_{p''}\in P$ and $0<\alpha<1$, then $\boldsymbol{x}_p=\boldsymbol{x}_{p'}=\boldsymbol{x}_{p''}$
- The corresponding vector $\boldsymbol{x}_f$ is a bfs of F

Optimality

1. For any instance of LP an optimal bfs exists, i.e. there is an optimal vertex of P.)

Proof: When $\boldsymbol{x}_o$ is the optimal solution and vertex *j* has the lowest cost $\boldsymbol{d}^T\boldsymbol{x}_j$

$$\boldsymbol{d}^T\boldsymbol{x}_{\boldsymbol{o}}=\sum_{i=1}^{N}\alpha_i\boldsymbol{d}^T\boldsymbol{x}_{\boldsymbol{i}}\geq\boldsymbol{d}^T\boldsymbol{x}_{\boldsymbol{j}}\sum_{i=1}^{N}\alpha_i=\boldsymbol{d}^T\boldsymbol{x}_{\boldsymbol{j}}$$

2. If *q* bfs's of F or *q* vertices of P are optimal, their convex combinations are optimal.

Summary

• LP can be though of as a convex polytope P.
• LP has at least one optimal bfs.
• The optimal bfs is a vertex of the polytope P.

What does this mean?

• The optimal solution for any LP can be found at the vertices of the corresponding polytope P.
• LP can be solved in a finite number of steps!