T-61.140 Signal Processing Systems

Exercise material for autumn 2003 - Solutions. Some formulae from Page 66.

1. Basics of complex numbers:

   a) $\frac{1}{2}e^{-j\pi} = \frac{1}{2}\cdot(-1) = -\frac{1}{2}$

   b) $e^{j5\pi/2} = e^{j(2\pi+\frac{\pi}{2})} = \underbrace{e^{j2\pi}}_{=1}\cdot e^{j\frac{\pi}{2}} = j$

   c) $-2 = 2\,e^{j\pi}$

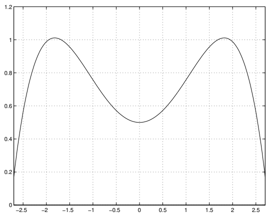
   d) $1+j = \sqrt{1^2+1^2}\,e^{\frac{\pi}{4}} = \sqrt{2}\,e^{\frac{\pi}{4}}$

   e) $zz^* = r\,e^{j\theta}r\,e^{-j\theta} = r^2\,e^{j(\theta-\theta)} = r^2$

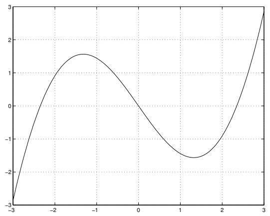
   f) $(z_1+z_2)^* = (x_1+jy_1+x_2+jy_2)^* = (x_1+x_2+j(y_1+y_2))^* =$
   $x_1+x_2-j(y_1+y_2) = x_1-jy_1+x_2-jy_2 = z_1^*+z_2^*$

2. Even and odd functions:

   The function $f(x)$ is even if $f(-x)=f(x)$.
   The function $f(x)$ is odd if $f(-x)=-f(x)$.



An even function.



An odd function.

Any function $f(x)$ can be expressed as a sum of its even and odd components:

$$\begin{aligned} f(x) &= \mathcal{E}ven\{f(x)\} + \mathcal{O}dd\{f(x)\}\text{ , where} \\ \mathcal{E}ven\{f(x)\} &= 1/2[f(x)+f(-x)] \\ \mathcal{O}dd\{f(x)\} &= 1/2[f(x)-f(-x)] \end{aligned}$$

a)

$$\begin{aligned} H(\omega) &= \mathcal{E}ven\{e^{j\omega}\} = \frac{1}{2}(e^{j\omega}-e^{-j\omega}) = \frac{1}{2}(\cos\omega + j\sin\omega + \cos(-\omega) + j\sin(-\omega)) \\ &= \frac{1}{2}(\cos\omega + j\sin\omega + \cos\omega - j\sin\omega) = \cos\omega \end{aligned}$$

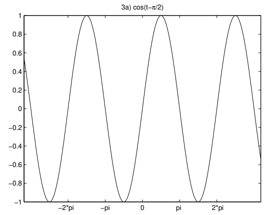
b)

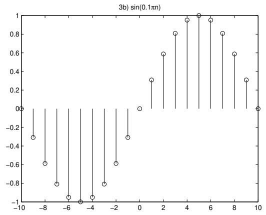
$$\begin{aligned} y(t) &= \mathcal{O}dd\{\sin(4\pi t)u(t)\} = \frac{1}{2}[\sin(4\pi t)u(t) - \sin(-4\pi t)u(-t)] \\ &= \frac{1}{2}[\sin(4\pi t)u(t) + \sin(4\pi t)u(-t)] = \frac{1}{2}\sin(4\pi t)[u(t)+u(-t)] \\ &= \frac{1}{2}\sin(4\pi t) \end{aligned}$$

3. Sketch the following signals and sequences around origo ($t=0$ *or* $n=0$).

   a) $x_1(t) = \cos(t-\pi/2)$

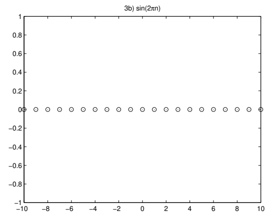
   b) $x_2[n] = \sin(0.1\pi n)$

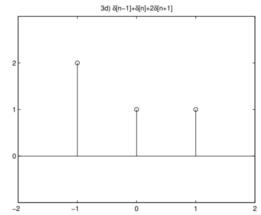




   c) $x_3[n] = \sin(2\pi n)$

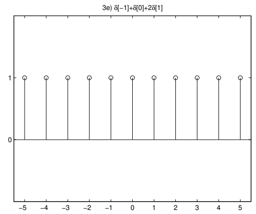
   d) $x_4[n] = \delta[n-1]+\delta[n]+2\delta[n+1]$

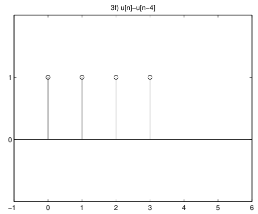




   e) $x_5[n] = \delta[-1]+\delta[0]+2\delta[1] = 0+1+0 = 1$

   f) $x_6[n] = u[n]-u[n-4]$





4. Which of the following continuous-time signals are periodic? Derive the fundamental period of periodic signals.

For a periodic continuous-time signal, $x(t) = x(t+T)$, where T is the fundamental period.

$$\begin{aligned} a)\ x(t) &= 3\cos(\frac{8\pi}{31}t) = 3\cos(\frac{8\pi}{31}(t+T)) = \\ &3\cos(\frac{8\pi}{31}t + \frac{8\pi}{31}T) = \\ &3\cos(\frac{8\pi}{31}t + 2\pi(\frac{4}{31}T)) \text{ is periodic, the fundamental period is } 31/4 \end{aligned}$$

$b)$ $x(t) = \exp(j(\pi t - 1))$ is periodic, the fundamental period is 2

$$\begin{aligned} c)\ x(t) &= \cos(\frac{\pi}{8}t^2) = \cos(\frac{\pi}{8}(t+T)^2) = \\ &\cos(\frac{\pi}{8}t^2 + \frac{\pi}{8}(2tT + T^2)) = \\ &\cos(\frac{\pi}{8}t^2 + 2\pi(\frac{tT}{8} + \frac{T^2}{16})) \text{ is not periodic} \end{aligned}$$

In (c) there is no any constant T for which the latter term would be 2π-multiple. Because of t it is changing all the time. Plot the function!

5. Which of the following discrete-time sequences are periodic? Derive the fundamental period of periodic sequences.

For a periodic discrete-time sequence, $x[n] = x[n+N]$, where $N \in \mathbb{Z}$ is the fundamental period.

$$\begin{aligned} a)\ x[n] &= 3\cos(\frac{8\pi}{31}n) = 3\cos(\frac{8\pi}{31}(n+k)) = \\ &3\cos(\frac{8\pi}{31}n + \frac{8\pi}{31}k) = \\ &3\cos(\frac{8\pi}{31}n + 2\pi(\frac{4}{31}k)) \end{aligned}$$

is periodic, the smallest possible k for which $\frac{4}{31}k \in \mathbb{Z}$,

that is the fundamental period, is 31

$b)$ $x[n] = \cos(\frac{n}{8} - \pi)$

is not periodic as no multiply of 16π is an integer

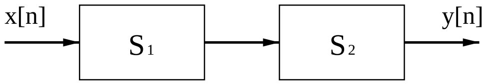
$c)$ $x[n] = 2\cos(\frac{\pi}{4}n) + \sin(\frac{\pi}{8}n) - 2\cos(\frac{\pi}{2}n + \frac{\pi}{6})$

is periodic. The fundamental period is the smallest common numerator, i.e. 16

6. Consider two systems S_1, S_2, whose input-outputrelations are:

$$\begin{aligned} S_1 &: y[n] = x[n] + 2x[n-2] \\ S_2 &: y[n] = x[n] - 3x[n-1] - 2x[n-2] \end{aligned}$$

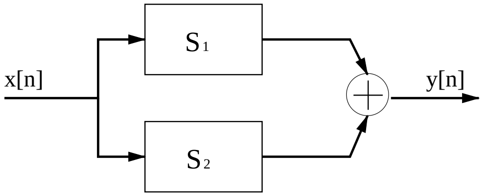
a) S_1 and S_2 in series Output can be found by setting the output of the previous system as an input to the next one.



S_1:n jälkeen $y_1[n] = x[n] + 2x[n-2]$, joka sijoitetaan S_2:en:
$y_2[n] = y_1[n] - 3y_1[n-1] - 2y_1[n-2] =$
$(x[n] + 2x[n-2]) - 3(x[n-1] + 2x[n-3]) - 2(x[n-2] + 2x[n-4]) =$
$x[n] - 3x[n-1] - 6x[n-3] - 4x[n-4]$

With Matlab `conv`:
```
conv([1 0 2],[1 -3 -2])
ans = [1 -3 0 -6 -4]
```

b) S_1 and S_2 in parallel The output can be found by adding the two lines.



$y[n] = (x[n] + 2x[n-2]) + (x[n] - 3x[n-1] - 2x[n-2]) =$
$2x[n] - 3x[n-1]$

7. The output of the system is

$$y[n] = \frac{x[n+1] - x[-n-1]}{2}.$$

a) The system is **memoryless** (book p.44), if the output depends only on the values of the input at the very same time. When examing the system S, one can see $x[-n-1]$, so S has memory.

b) The system is **linear** (p. 53), if it is additive ($x_1(t) + x_2(t) \ \rightarrow\ y_1(t) + y_2(t)$) and scalable ($ax_1(t) \ \rightarrow\ ay_1(t)$).

The outputs for x_1 and x_2 are known:

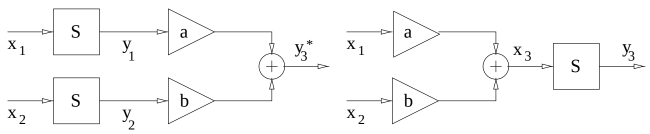
$y_1[n] = 0.5\,(x_1[n+1] - x_1[-n-1])$
$y_2[n] = 0.5\,(x_2[n+1] - x_2[-n-1])$

Examine the linear property by feeding the input

$x_3[n] = \alpha x_1[n] + \beta x_2[n]$

We get:

$$\begin{aligned} y_3[n] &= \frac{1}{2}\,(x_3[n+1] - x_3[-n-1]) \\ &= \frac{1}{2}\,((\alpha x_1[n+1] + \beta x_2[n+1]) - (\alpha x_1[-n-1] + \beta x_2[-n-1])) \\ &= \alpha\left(\frac{1}{2}\,(x_1[n+1] - x_1[-n-1])\right) + \beta\left(\frac{1}{2}\,(x_2[n+1] - x_2[-n-1])\right) \\ &= \alpha y_1[n] + \beta y_2[n] = y_3[n]* \end{aligned}$$



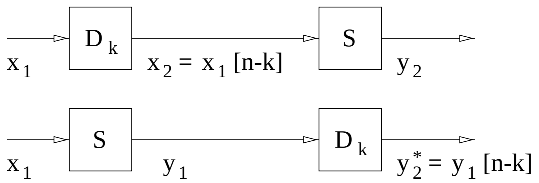
The system is therefore linear.

c) A system is **time-invariant** (p.50), if the behaviour of the system doen't depend on time. Examine with a delayed input

$$x_2[n] = x_1[n-k].$$

If the system is time-invariant, should

$$y_2[n] = y_1[n-k].$$



Examine, if this is true in this case.

$$\begin{aligned} y_2[n] &= \frac{x_2[n+1] - x_2[-n-1]}{2} \\ &= \frac{x_1[(n+1)-k] - x_1[(-n-1)-k]}{2} \\ y_1[n-k] &= \frac{x_1[(n-k)+1] - x_1[-(n-k)-1]}{2}. \end{aligned}$$

Because $y_2[n] \neq y_1[n-k]$, the system is not time-invariant.

d) The system is **stable** (p. 48), when the system gives a bounded output always when the input is bounded. Suppose that it is known for the input of the systems

$$\max|x[n]| \leq B\ \forall\ n.$$

So, (using triangle equation $|a-b| \leq |a| + |b|$)

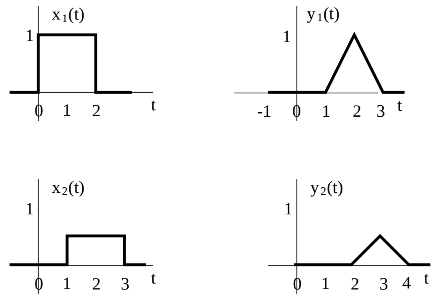
$$\max|y[n]| = \left|\frac{x[n+1] - x[-n-1]}{2}\right| \leq \frac{|x[n+1]| + |x[-n-1]|}{2} \leq \frac{B+B}{2} = B.$$

The system is stable.

e) The system is **causal** (p. 46), when the output $y(t)$ depends only on the input $x(t)$ of the time moment t and the past. However, now the output of the system S depends on the term $x[n+1]$, so S is not causal.

8. Using the properties of linear, time-invariant systems (additivity, scaling, shifting in time) it can be observed that signal $x_2(t)$ can be formed by scaling and delaying $x_1(t)$: $x_2(t) = 0.5\,x_1(t-1)$, so the output $y_2(t)$ is $y_2(t) = 0.5\,y_1(t-1)$.

   See the figure below. x_2 can be contructed also as a superposition from several scaled and delayed x_1.



9. Let's calculate convolution sum when

   a) $x[n]$ and $h[n]$ are like in the figure 15. Look at the figure 16. $-k$ in the impulse response $h[n-k]$ rotates the time axis whereas n shifts, in the example figure 16 $n = 3$. For each n the convolution is calculated with the dot product of $x[k]$ and $h[3-k]$, and summed together. The result for an input $x[n]$ is in the figure 17.

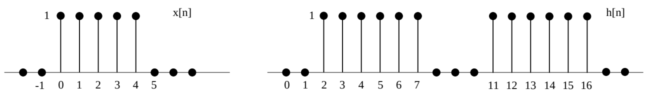


Figure 15: *Problem 9: Input and impulse response.*

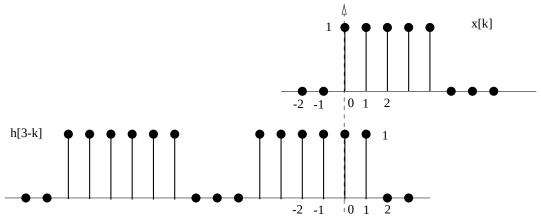


Figure 16: *Problem 9: Convolution when $n = 3$. The input and impulse response are nonzero only when $k = 0$ and $k = 1$, producing the value 2.*

b) $x[n] = \alpha^n u[n]$
$h[n] = \beta^n u[n]$.

Remember that the sum of geometric series, where ratio is $| q < 1|$

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

The convolution

$$y[n] = \sum_{k=-\infty}^{\infty} \alpha^k u[k] \beta^{n-k} u[n-k].$$

Examining the step function we can notice that $u[k] = 1$, when $k \geq 0$, and $u[n-k] = 1$, when $k \leq n$. The sum is nonzero when $x \in [0, n]$, that is

$$y[n] = \sum_{k=0}^{n} \alpha^k \beta^{n-k} = \beta^n \sum_{k=0}^{n} \alpha^k \beta^{-k}.$$

Now we have $q = \frac{\alpha}{\beta}$. Remember that $|q < 1|$ when $n \to \infty$. For series

$$y[n] = \beta^n \frac{1 - \left[\frac{\alpha}{\beta}\right]^{n+1}}{1 - \frac{\alpha}{\beta}} = \frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha}.$$

c) $x[n] = (-\frac{1}{2})^n u[n-4]$
$h[n] = 4^n(2-n)$.

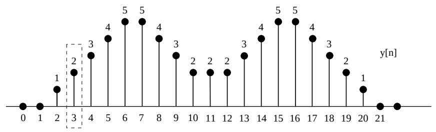


Figure 17: *Problem 9: The result of the convolution, now at $n = 3$: $y[3] = 2$*

Because of the step function $x[k]$ is zero, when $k < 4$. So the output $y[n]$ is

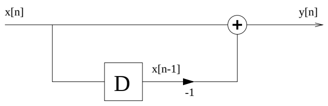
$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} (-\frac{1}{2})^k u[k-4] 4^{n-k} (2-(n-k)) \\ &= \sum_{k=4}^{\infty} (-1)^k 2^{-k} 4^{n-k} (2-n+k) \\ &= \sum_{k=4}^{\infty} (-1)^k 2^{2n-3k} (2-n+k) \end{aligned}$$

10. (a) yes/yes, (b) no/no, (c) no/yes, (d) no/no, (e) no/yes, (f) no/yes.

11. Consider a system defined by the difference equation

$$y[n] = x[n] - x[n-1] \ .$$

a) The block diagram of the system (D denotes a delay unit):



b) The impulse response $h[n]$ is defined as the response of the system to the unit impulse $\delta[n]$:

$$h[n] = \delta[n] - \delta[n-1]$$

c) The response of the system to input sequence $x[n] = \left(\frac{1}{3}\right)^n u[n]$ in Figure 18.

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= \left(\frac{1}{3}\right)^n u[n] - \left(\frac{1}{3}\right)^{n-1} u[n-1] \\ &= \left(\frac{1}{3}\right)^{n-1} \left[\frac{1}{3} u[n] - u[n-1]\right] \end{aligned}$$

12. Consider a system defined by the difference equation

$$y[n] - \frac{1}{2} y[n-1] = x[n] \ .$$

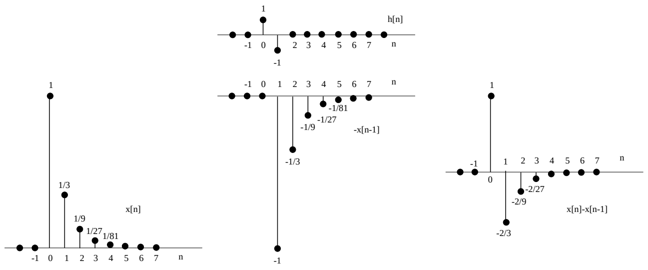


Figure 18: *Problem 11(c): left: input* $x[n]$*, middle top* $h[n]$*, middle bottom* $-x[n-1]$*, right:* $y[n] = x[n] - x[n-1]$.
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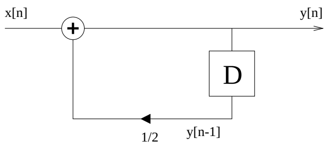


a) The block diagram of the system (D denotes a delay unit) above. Notice the feedback loop!!!

b) The impulse response $h[n]$ is the response to the unit impulse $\delta[n]$:

$$\begin{aligned}
n = 0 \quad &: \quad x[0] = 1 \quad y[0] = \frac{1}{2}y[-1] + x[0] = 1 \\
n = 1 \quad &: \quad x[1] = 0 \quad y[1] = \frac{1}{2}y[0] + x[1] = \frac{1}{2} \\
n = 2 \quad &: \quad x[2] = 0 \quad y[2] = \frac{1}{2}y[1] + x[2] = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \\
n = 3 \quad &: \quad x[3] = 0 \quad y[3] = \frac{1}{2}y[2] + x[3] = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \\
n = 4 \quad &: \quad x[4] = 0 \quad y[4] = \frac{1}{2}y[3] + x[4] = \left(\frac{1}{2}\right)^4
\end{aligned}$$

From above, it can be seen that $h[n] = \left(\frac{1}{2}\right)^n u[n]$.

c) Difference equations are solved in two parts. First, we solve the so called *homogeneous equation*, in which the input sequence $x[n]$ is set to zero. A sequence satisfying the homogeneous equation is $y[n] = z^n$. The solution to the *particular equation* is then typically found using a sequence having the same form as the input sequence.

The complete solution to the equation is a linear combination of these two solutions

$$y[n] = Ay_h[n] + By_p[n],$$

where A is determined from some auxiliary condition and B from the solution to the particular equation. Let us proceed as described above in solving the exercise:

H.E.

$$\begin{aligned}
y[n] - \frac{1}{2}y[n-1] &= 0 \ , \ y = z^n \\
\Rightarrow z^n - \frac{1}{2}z^{n-1} &= 0 \ | \cdot z^{-n+1} \\
\Leftrightarrow z &=
\end{aligned}$$

The solution of the homogeneous equation is thus

$$y_h[n] = A\left(\frac{1}{2}\right)^n.$$

P.E. Let us consider a situation with $n > 0$.

$$\begin{aligned}
y[n] - \frac{1}{2}y[n-1] &= x[n] \ , \ y = Bx[n] = B\left(\frac{1}{3}\right)^n \\
\Rightarrow B\left(\frac{1}{3}\right)^n - \frac{1}{2}B\left(\frac{1}{3}\right)^{n-1} &= \left(\frac{1}{3}\right)^n \qquad | \cdot \left(\frac{1}{3}\right)^{-n} \\
\Leftrightarrow B - \frac{3}{2}B = 1 \Leftrightarrow B &= -2.
\end{aligned}$$

The complete solution is now of form

$$y[n] = A\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n.$$

The examined system is causal as it does not require any future values to determine the current values. Therefore, the response $y[n] = 0$ when $n < 0$, since the input is $x[n] = 0$ when $n < 0$. Based on this auxiliary condition we get

$$y[0] - \frac{1}{2}y[-1] = x[0],$$

where $y[-1] = 0$. Let us substitute here the complete solution:

$$\begin{aligned}
A\left(\frac{1}{2}\right)^0 - 2\left(\frac{1}{3}\right)^0 &= \left(\frac{1}{3}\right)^0 \\
A &= 3,
\end{aligned}$$

so the complete solution becomes:

$$y[n] = \begin{cases} 3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}.$$

Note that the exercise can be solved (1) by z-transforming both x and y, using partial fraction decomposition to solve the coefficients and then performing an inverse z-transform or (2) by using convolution $x[n] * h[n]$.

13. It follows from the associative property of LTI systems (Oppenheim p. 107) that

$$h_1[n] * (h_2[n] * h_2[n]) = (h_2[n] * h_2[n]) * h_1[n].$$

Since the impulse response $h_2[n]$ is given in the exercise, the remaining task is to determine $h_1[n]$. This process is called *deconvolution*.

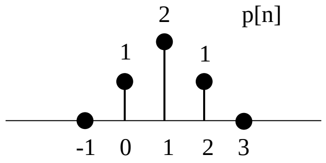
Let us first calculate the convolution $p[n] = h_2[n] * h_2[n]$

$$p[n] = \sum_{k=-\infty}^{\infty} h_2[k]h_2[n-k] = \sum_{k=-\infty}^{\infty} (u[k]-u[k-2])(u[n-k]-u[n-k-2]).$$

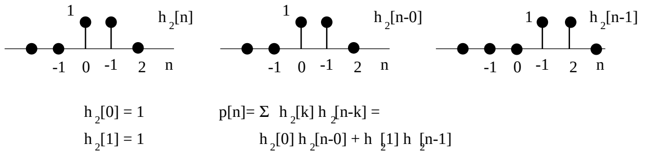
The term $u[k]-u[k-2]$ in the convolution has non-zero values only when $k=0$ or $k=1$. Therefore, the convolution can be written as

$$p[n] = \sum_{k=0}^{1} u[n-k]-u[n-k-2] = (u[n-0]-u[n-2]) + (u[n-1]-u[n-3])\,.$$

The impulse response is $p[0]=1$, $p[1]=2$, $p[2]=1$, and $p[k]=0$, otherwise. The impulse response of the system $h_2[n] * h_2[n]$ is thus:



The same thing graphically:



a) The impulse response $h_1[n]$ is to be determined with *deconvolution*. First, it is known that the length of the non-zero portion of the impulse response $h_1[n]$ must be $N_1 = N + 1 - N_2 = 7 + 1 - 3 = 5$.
   Now, the exercise can be solved by considering $p[n]$ as the input to the system h_1

$$h[n] = \sum_{k=0}^{2} p[k]h_1[n-k],$$

   where we have to find a time-invariant impulse response so that

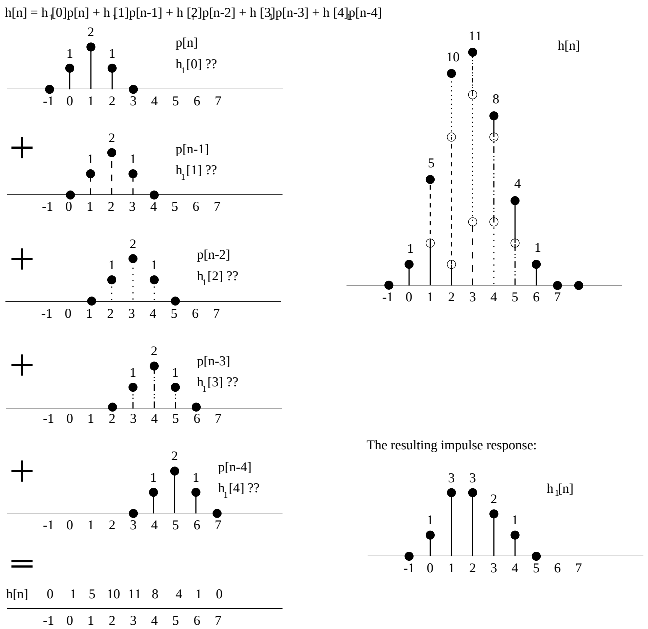
$$1 \cdot h_1[0] + 2 \cdot h_1[n-1] + 1 \cdot h_1[n-2] = h[n],$$

where the scaling coefficients originate from $p[n]$.

Another way is to find such scaling coefficients that the impulse response $h[n]$ can be expressed as superpositions of $p[n]$. That is, the unknown impulse response of the system h_1 is considered as the input to the system $h_2[n] * h_2[n]$. Let us solve the exercise in this latter way. In the figure on the next page, the construction of $h[n]$ from several time-shifted and multiplied versions of $p[n]$ is illustrated.

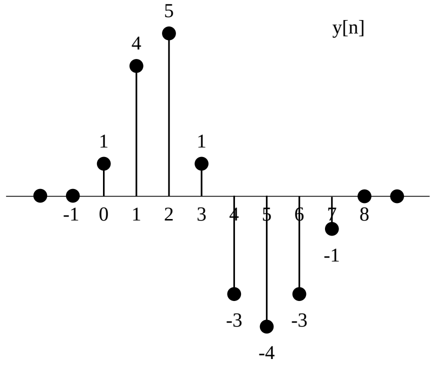
The impulse response of the whole system can thus be expressed as follows (compare with the figure on the next page):

$$\begin{aligned} h[n] &= h_1[0]p[n] + h_1[1]p[n-1] + h_1[2]p[n-2] + h_1[3]p[n-3] + h_1[4]p[n-4] \\ &= p[n] + 3p[n-1] + 3p[n-2] + 2p[n-3] + p[n-4] \end{aligned}$$



b) The desired response is a superposition of two impulse responses, where a time-shifted (with one unit) and inverted version of the impulse response is subtracted from the original impulse response $h[n]$.

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[0]h[n-0] + x[1]h[n-1] = h[n] - h[n-1]$



14. First, let us use the definition of convolution directly. Note, however, that $x[n] = 3^{-|n|}$ (check this yourself!).

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_k x[k]h[n-k] \\ &= \sum_k 3^{-|k|}\left(\frac{1}{4}\right)^{n-k} u[n-k+3] \\ &= \sum_{k=-\infty}^{n+3} 3^{-|k|}(1/4)^{n-k} \end{aligned}$$

Then, let us use the distributive property of convolution (Oppenheim p. 104):

$$\begin{aligned} y[n] &= 3^n u[-n-1] * (1/4)^n u[n+3] && +(1/3)^n u[n] * (1/4)^n u[n+3] \\ &= \sum_{k=-\infty}^{\infty} 3^k u[-k-1]\left(\frac{1}{4}\right)^{n-k} u[n+3-k] && + \sum_{k=-\infty}^{\infty}\left(\frac{1}{3}\right)^k u[k]\left(\frac{1}{4}\right)^{n-k} u[n+3-k] \\ &= \sum_{k=-\infty}^{min(-1,n+3)} 3^{-|k|}\left(\frac{1}{4}\right)^{n-k} && + \sum_{k=0}^{n+3} 3^{-|k|}\left(\frac{1}{4}\right)^{n-k} \end{aligned}$$

Now, if $n+3 > -1$, we get:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{-1} 3^{-|k|}\left(\frac{1}{4}\right)^{n-k} + \sum_{k=0}^{n+3} 3^{-|k|}\left(\frac{1}{4}\right)^{n-k} \\ &= \sum_{k=-\infty}^{n+3} 3^{-|k|}(1/4)^{n-k} \end{aligned}$$

If $n+3 < -1$, the latter sum term equals zero and the former sum term equals

$$y[n] = \sum_{k=-\infty}^{n+3} 3^{-|k|}(1/4)^{n-k}.$$

So, it can be concluded that the result is the same than without using the distributive property.

15. a) Let us verify that we get the difference equation

$$2y[n] - y[n-1] + y[n-3] = x[n] - 5x[n-4],$$

by considering a cascade of two systems S_1 and S_2 where

$$\begin{aligned} S_1 :\ & 2y_1[n] = x_1[n] - 5x_1[n-4], \\ S_2 :\ & y_2[n] = \frac{1}{2}y_2[n-1] - \frac{1}{2}y_2[n-3] + x_2[n]. \end{aligned}$$

After S_1, the response is thus

$$S_1 :\ y_1[n] = \frac{1}{2}x[n] - \frac{5}{2}x[n-4]$$

and as this in fed as the input to the system S_2, we get

$$\begin{aligned} y[n] &= \frac{1}{2}y[n-1] - \frac{1}{2}y[n-3] + \frac{1}{2}x[n] - \frac{5}{2}x[n-4] \\ \Leftrightarrow 2y[n] - y[n-1] &+ y[n-3] = x[n] - 5x[n-4], \end{aligned}$$

which was to be established. The block diagrams are shown in Figure 19.

16. LTI-system changes only the amplitude of different frequencies, not the frequency itself, see the section 3.2., p. 182:

$$e^{j\omega n} \rightarrow H(e^{j\omega})\, e^{j\omega n}$$

So, S_1 cannot be LTI, but S_2 can be.

17. Synthesis equation is:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t},$$

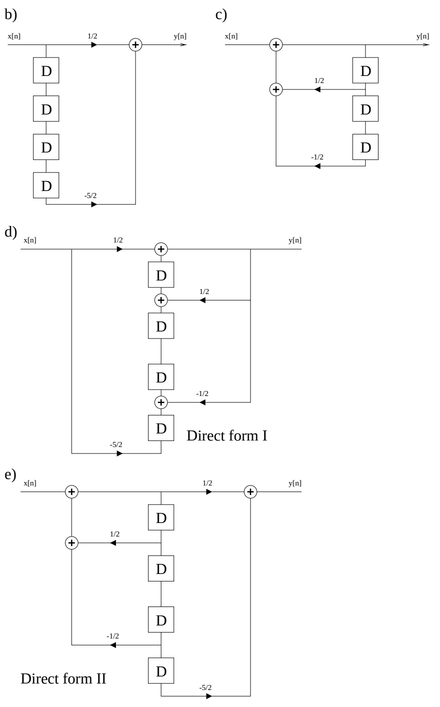


Figure 19: *Block diagrams for Problem 15.*

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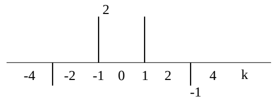


Now the coefficients a_k are shown above.

As the length of basic period is $T = 4$, so $f_0 = 1/4$ and $\omega_0 = 2\pi f_0 = 0.5\pi$.

We get the signal shown in Figure 20:

$$\begin{aligned} x(t) &= a_{-3}\ e^{j(-3)\omega_0 t} + a_{-1}\ e^{j(-1)\omega_0 t} + a_1\ e^{j(1)\omega_0 t} + a_3\ e^{j(3)\omega_0 t} \\ &= -1e^{j(-3)(0.5\pi)t} + 2e^{j(-1)(0.5\pi)t} + 2e^{j(1)(0.5\pi)t} - 1e^{j(3)(0.5\pi)t} \\ &= -(e^{-j(1.5\pi)t} + e^{j(1.5\pi)t}) + 2(e^{j(-0.5\pi)t} + e^{j(0.5\pi)t}) \\ &= -2\,cos(1.5\pi t) + 4\,cos(0.5\pi t) \end{aligned}$$

18. Answer can be seen right from the series representation (synthesis equation).

   a) $x(t) = ... + a_{-1}e^{j(-1)\omega_0 t} + ...,$

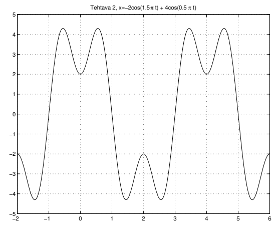


Figure 20: *Problem 16: signal.*

from which $a_{-1} = 1$ and for all other k $a_k = 0$.
Otherwise, analysis equation can also be used

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}dt$$

For example, a_{-1}:

$$a_{-1} = \frac{1}{T}\int_T x(t)e^{-j(-1)\omega_0 t}dt = \frac{1}{T}\int_T e^{-j\omega_0 t}e^{j\omega_0 t}dt = \frac{1}{T}\int_T 1\ dt = 1$$

With other k the integrand will have $2m\pi$ in its exponential term, so the integral will be zero over a period.

b) One has to calcutate the basic angular frequency. For the first one the period is $T_1 = 1$ and for the other $T_2 = 2/3$. So, the basic period is $T_0 = 2$, and $\omega_0 = 2\pi/2 = \pi$.
Using Euler's formula and basic angular frequency $\omega_0 = \pi$ we get:
$cos(2\pi t) = cos((2)(\pi)t) = 0.5\left(e^{j(2)(\pi)t} + e^{j(-2)(\pi)t}\right)$
and correspondingly with $cos(3\pi t)$, which lead

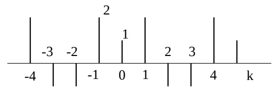
$$x(t) = a_{-3}e^{j(-3)(\pi)t} + a_{-2}e^{j(-2)(\pi)t} + a_2 e^{j(2)(\pi)t} + a_3 e^{j(3)(\pi)t}$$

from which
$a_{-3} = a_3 = 1/2,\ a_{-2} = a_2 = 1/2,$ otherwise $a_k = 0$.

19. Synthesis equation is

$$x[n] = \sum_{k=\langle N\rangle} a_k\, e^{jk\omega_0 n}$$

Now the Fourier coefficients a_k are:



Notice that **Fourier coefficients of discrete signal are periodic**, in other words, for example $a_0 = a_5 = a_{-5} = a_{10} = a_{-10} = ...$ Notation: $k = \langle 0 \rangle$ means $k = 5m$, where $m \in \mathbb{Z}$ and 5 is length of period.

As the period is 5, then $f_0 = 1/5$ and $\omega_0 = 2\pi f_0 = 0.4\pi$.

So we get a sequence in Figure 21:

$$\begin{aligned} x[n] &= -1e^{j(-2)(0.4\pi)n} + 2e^{j(-1)(0.4\pi)n} + 1e^0 + 2e^{j(1)(0.4\pi)n} - 1e^{j(2)(0.4\pi)n} \\ &= -(e^{-j(0.8\pi)n} + e^{j(0.8\pi)n}) + 1 + 2(e^{j(-0.4\pi)n} + e^{j(0.4\pi)n}) \\ &= -2\,cos(0.8\pi n) + 4\,cos(0.4\pi n) + 1 \end{aligned}$$

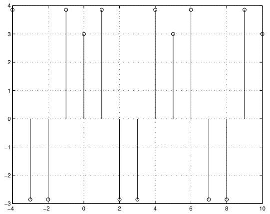


Figure 21: *Problem 19:* $-2\cos(0.8\pi n) + 4\cos(0.4\pi n) + 1$.

20. Using series expansion

   a) Basic angular frequency $\omega_0 = \pi/3$ and the length of period is $N = 6$:

   $$x_1[n] = ... + a_{-1}e^{j(-1)(\pi/3)n} + a_0 + a_1 e^{j(1)(\pi/3)n} + ...$$

   where $a_{-1} = a_1 = 1/2$, and other $a_k = 0$, when $k = \langle -2, 0, 2, 3 \rangle$.

   b) The period of the first term is $N_1 = 4$ and of the second $N_2 = 8$. So, the period of the combination is $N = 8$ and angular frequency $\omega_0 = \pi/4$. Therefore,

   $$x[n] = ... + a_{-2}e^{j(-2)(\pi/4)n} + a_{-1}e^{j(-1)(\pi/4)n} + a_0 + a_1 e^{j(1)(\pi/4)n} + a_2 e^{j(2)(\pi/4)n} + ...$$

   from which $a_{-2} = \frac{-1}{2j}, a_{-1} = \frac{1}{2}, a_1 = \frac{1}{2}, a_2 = \frac{1}{2j}$, and other $a_k = 0$, when $k = \langle -3, 0, 3, 4 \rangle$.

21. A periodic signal $x(t)$

   a) a_0 gives the mean of the signal, DC-component.

   $$a_0 = \frac{1}{T}\int_T x(t)dt = 1/2\int_0^2 x(t)dt = \frac{1}{2}$$

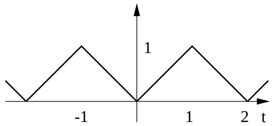


Figure 22: Original $x(t)$, F-coefficients a_k

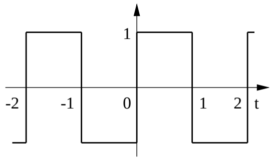


Figure 23: Derivate of $x(t)$, F-coefficients b_k

   b) Derivate of the signal is

   $$dx(t)/dt = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$$

   See the Example 3.5 and Example 3.6 from the book. There we have a formula 3.44, which is also here below. It gives the Fourier-coefficients, where T_1 is the moment where step goes from one to zero, and T is length of period, look at the picture. The coefficients are:

   $$d_k = \frac{2\,sin(k\omega_0 T_1)}{k\omega_0 T}, k \neq 0$$

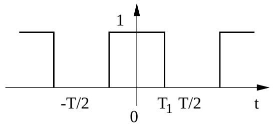


Figure 24: From book, F-coefficients d_k

   From the table 3.1 it is seen that when a signal, whose Fourier-coefficients are a_k, is delayed by t_0 in time domain, then Fourier-coefficients of delayed signal are $a_k e^{-jk\omega_0 t_0}$.

   In this exercise the mean of the (derivate) signal is zero, that is $b_0 = 0$. The amplitude of the signal is double and $T_1 = 0.5$, $T = 2$, from which $\omega_0 = \frac{2\pi}{2} = \pi$. Signal is delayed by $t_0 = 0.5$.
   Now c_k are d_k multiplied by two, b_k (derivate) is shifted c_k.

$$\begin{aligned} c_k &= 2\, d_k \\ &= 2\,\frac{2\, sin(k\omega_0 T_1)}{k\omega_0 T} \\ &= 2\,\frac{2\, sin(k\pi/2)}{2\, k\pi} \\ &= 2\,\frac{sin(k\pi/2)}{k\pi} \end{aligned}$$

$$\begin{aligned} b_k &= c_k\; e^{-jk\omega_0 t_0} \\ &= c_k\; e^{-jk\pi/2} \\ &= 2\,\frac{sin(k\pi/2)}{k\pi}\; e^{-jk\pi/2} \end{aligned}$$

c) Derivation property means that if the Fourier-coefficients of the signal $x(t)$ are a_k, then those of derivate $dx(t)/dt$ are $jk\omega_0\; a_k$.

Let the Fourier-coefficients of the original signal be a_k. Solution from 6b) b_k are then $a_k\; jk\omega_0$. Because $T = 2$, then $\omega_0 = \pi$.

$$\begin{aligned} a_k\; jk\omega_0 &= 2\frac{\sin(k\pi/2)}{k\pi}\, e^{-jk\pi/2} \\ a_k &= \frac{1}{jk\omega_0}\; 2\frac{\sin(k\pi/2)}{k\pi}\, e^{-jk\pi/2} \\ &= \frac{-j}{k\pi}\; 2\frac{\sin(k\pi/2)}{k\pi}\, e^{-jk\pi/2} \\ &= (-j)\; 2\frac{\sin(k\pi/2)}{(k\pi)^2}\, e^{-jk\pi/2} \end{aligned}$$

The results from the 6c, Fourier-coefficients of the original signal:

$$a_k = \begin{cases} -j\; 2\frac{\sin(k\pi/2)}{(k\pi)^2}\, e^{-jk\pi/2}, & k \neq 0 \\ \frac{1}{2}, & k = 0 \end{cases}.$$

Coefficient could have been calculated straigth from the original signal by using F-series equations.

22. . Fourier-coefficients for discrete sequences

a) Corresponding example can be found from the book, example 3.13. Sequence $x[n]$ in Figure 25.

Find Fourier-coefficients a_k. Signal can be thought as a sum of three signals, $x_1[n]$, $x_2[n]$ ja $x_3[n]$ (look at the big figure), whose Fourier-coefficients are b_k, c_k and d_k, respectively. The period is $N = 6$ and $\omega_0 = \pi/3$. Now the Fourier-coefficients of $x[n]$ can be found as a sum $a_k = b_k + c_k + d_k$. Why?

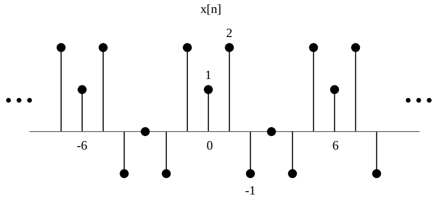


Figure 25: *Problem 22: Sequence.*

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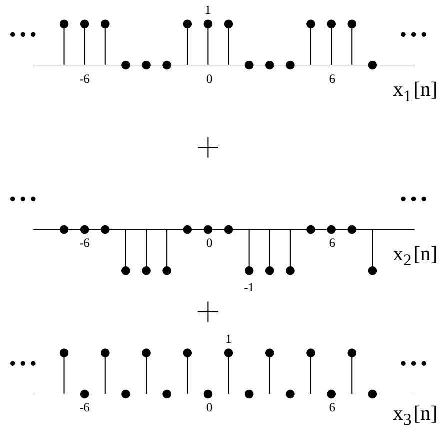


Figure 26: *Problem 22: Three subsignals form $x[n]$*

First, calculate coefficients of the 1st signal $x_1[n]$:

$$b_0 = \frac{1}{6}\sum_{n=-1}^{4} x[n] = \frac{1}{2},\; kun\; k = \langle 0 \rangle$$

and other coefficients (routine!)

$$b_k = \frac{1}{6}\sum_{n=-1}^{1} e^{-jk\frac{\pi}{3}n} = \frac{1}{6} e^{jk\frac{\pi}{3}} \sum_{n=0}^{2} e^{-jk\frac{\pi}{3}n} = \frac{1}{6}\frac{e^{jk\frac{\pi}{2}}}{e^{jk\frac{\pi}{6}}} \cdot \frac{1-e^{-jk\pi}}{1-e^{-\frac{jk\pi}{3}}} = \frac{1}{6}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})},$$

just as we wanted, like Example 3.12, formulae 3.104, 3.105).

b_2 and b_4 are always zero. So the coefficients b_k:

$$b_k = \begin{cases} \frac{1}{2}, & k = \langle 0 \rangle \\ \frac{1}{6}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})}, & k = \langle 1, 3, 5 \rangle \\ 0, & \text{else} \end{cases}.$$

The second pulse $x_2[n]$ is the same as the first, but shifted by three and negative

$$c_k = \frac{1}{6}\sum_{n=2}^{4}(-1)\,e^{-jk\frac{\pi}{3}n} = -\frac{1}{6}e^{-jk\frac{2\pi}{3}}\sum_{n=0}^{2}e^{-jk\frac{\pi}{3}n} = -\frac{1}{6}e^{-jk\frac{3\pi}{3}}\frac{e^{jk\frac{\pi}{2}}}{e^{jk\frac{\pi}{6}}}\cdot\frac{1-e^{-jk\pi}}{1-e^{-\frac{jk\pi}{3}}} = \frac{1}{6}(-1)^{k-1}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})}$$

Notice that $e^{-jk(\pi/3)3} = e^{-jk\pi} = (-1)^k$.

This can be resolved also with tables (book table 3.2 in page 221, Linearity $A = -1, a_k \to -a_k$, Time Shifting $n_0 = 3, a_k \to a_k\,e^{-jk\omega_0 n_0}$).

Coefficients are

$$c_k = \begin{cases} -\frac{1}{2} \\ \frac{1}{6}(-1)^{k-1}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})} \\ 0 \end{cases} = \begin{cases} -\frac{1}{2}, & k = \langle 0\rangle \\ \frac{1}{6}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})}, & k = \langle 1,3,5\rangle \\ 0, & \text{else} \end{cases}.$$

The third pulse $x_3[n]$

$$d_0 = \frac{1}{6}\sum_{n=0}^{5}x[n] = \frac{1}{2}$$

and other coefficients

$$d_k = \frac{1}{6}\sum x[n]e^{-jk\frac{\pi}{3}n},$$

$$d_k = \begin{cases} \frac{1}{6}(e^{-j\frac{\pi}{3}\cdot 1} + e^{-j\frac{\pi}{3}\cdot 3} + e^{-j\frac{\pi}{3}\cdot 5}) \\ \frac{1}{6}(e^{-j\frac{2\pi}{3}\cdot 1} + e^{-j\frac{2\pi}{3}\cdot 3} + e^{-j\frac{2\pi}{3}\cdot 5}) \\ \frac{1}{6}(e^{-j\pi\cdot 1} + e^{-j\pi\cdot 3} + e^{-j\pi\cdot 5}) \\ \frac{1}{6}(e^{-j\frac{4\pi}{3}\cdot 1} + e^{-j\frac{4\pi}{3}\cdot 3} + e^{-j\frac{4\pi}{3}\cdot 5}) \\ \frac{1}{6}(e^{-j\frac{5\pi}{3}\cdot 1} + e^{-j\frac{5\pi}{3}\cdot 3} + e^{-j\frac{5\pi}{3}\cdot 5}) \end{cases} = \begin{cases} \frac{1}{6}(\frac{1}{2} - \frac{\sqrt{3}}{2}j - 1 + \frac{1}{2} + \frac{\sqrt{3}}{2}j) = 0, & k = \langle 1\rangle \\ \frac{1}{6}(-\frac{1}{2} - \frac{\sqrt{3}}{2}j + 1 - \frac{1}{2} + \frac{\sqrt{3}}{2}j) = 0, & k = \langle 2\rangle \\ \frac{1}{6}(-1-1-1) = -1/2, & k = \langle 3\rangle \\ \frac{1}{6}(-\frac{1}{2} + \frac{\sqrt{3}}{2}j + 1 - \frac{1}{2} - \frac{\sqrt{3}}{2}j) = 0, & k = \langle 4\rangle \\ \frac{1}{6}(\frac{1}{2} + \frac{\sqrt{3}}{2}j - 1 + \frac{1}{2} - \frac{\sqrt{3}}{2}j) = 0, & k = \langle 5\rangle \end{cases}$$

where only $d_k = -\frac{1}{2}$ is non-zero when $k = \langle 3\rangle$

The result is the sum of three Fourier-coefficients:

$$a_k = b_k + c_k + d_k = \begin{cases} \frac{1}{2}, & k = \langle 0\rangle \\ \frac{1}{3}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})}, & k = \langle 1,5\rangle \\ 0, & k = \langle 2,4\rangle \\ -\frac{1}{2} + \frac{1}{3}\frac{\sin(\frac{\pi k}{2})}{\sin(\frac{\pi k}{6})}, & k = \langle 3\rangle \end{cases}$$

The solution using Matlab

```
x = [1 2 -1 0 -1 2]'
a = 1/length(x) * fft(x)
```

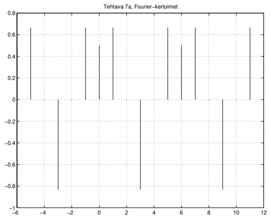



Figure 27: *Fourier-coefficients.*

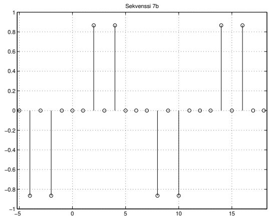


Figure 28: *Sequence.*

b) From the table 3.2 (Multiplication) it is seen that the exercise can be done as a convolution of coefficients. The period of the system is $N = 12$, $\omega_0 = \pi/6$.

First, Fourier-coefficients of $\sin(2\pi n/3)$.

$$b_0 = \frac{1}{12}\sum_{n=0}^{11}x[n] = 0.$$

The rest b_k:t

$$b_k = \frac{1}{12}\sum_{n=0}^{11}x[n]e^{-jk\frac{\pi}{6}n}$$

which results

$$b_k = \begin{cases} -\frac{1}{2}j, & k = \langle 4\rangle \\ \frac{1}{2}j, & k = \langle 8\rangle \\ 0 & k = \langle 0,1,2,3,5,6,7,9,10,11\rangle \end{cases}$$

Because it is sine signal, which is odd function, also the Fourier-coefficients form an odd function.n This can be seen $k < 0$, whose values are $-b_k$ compared to corresponding positive k.

Second, Fourier-coefficients of $cos(\pi n/2)$

$$c_k = \begin{cases} \frac{1}{2}, & k = \langle 3,9\rangle \\ 0 & k = \langle 0,1,2,4,5,6,7,8,10,11\rangle \end{cases}.$$

Calculating the convolution $a_k = \sum_{l=<N>} b_l c_{k-l}$:

$$a_k = \begin{cases} -\frac{1}{4}j, & k = \langle 1, 7 \rangle \\ \frac{1}{4}j, & k = \langle 5, 11 \rangle \\ 0, & k = \langle 0, 2, 3, 4, 6, 8, 9, 10 \rangle \end{cases}$$

Because there is a sine function (odd), so also the F-coefficients are odd:

$$a_k = \begin{cases} \frac{1}{4}j, & k = \langle 1, 7 \rangle \\ -\frac{1}{4}j, & k = \langle 5, 11 \rangle \\ 0, & k = \langle 0, 2, 3, 4, 6, 8, 9, 10 \rangle \end{cases}$$

We could also think that we will copy the range 0..11 to the place $-12..-1$ etc.

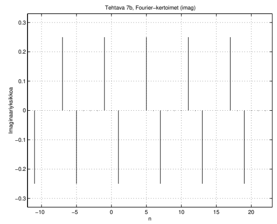


Figure 29: *Fourier-coefficients.*

Solution using Matlab:

```
n = [0:11]'
x = sin(2*pi*n/3) .* cos(pi*n/2)
a = 1/length(x) * fft(x)
```

23. Since the basic period of $x(t)$ is $T = 6$, the Fourier series representation of $x(t)$ can be written as

$$x(t) = \sum_k a_k \exp(jtk\pi/3)$$

As the signal $x(t)$ is real, it follows that $|a_k| = |a_{-k}|$. Furthermore, since $a_k = 0$ with $k = 0, 3, 4, 5, \ldots$, $a_k \neq 0$ only, when $k = -2, -1, 1, 2$. Now,

$$x(t) = a_2 \exp(jt2\pi/3) + a_2^* \exp(-jt2\pi/3) + a_1 \exp(jt\pi/3) + a_1^* \exp(-jt\pi/3)$$

Since $a_1 > 0$ and real,

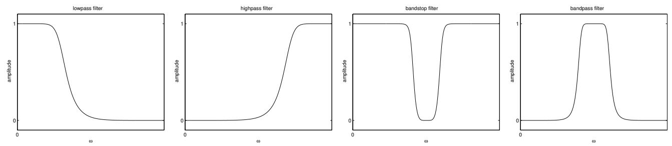
$$x(t) = a_1 2\cos(t\pi/3) + a_2 \exp(jt2\pi/3) + a_2^* \exp(-jt2\pi/3)$$

From the equation $x(t) = -x(t-3)$, it follows that

$$\begin{aligned} x(t) &= a_1 2\cos(t\pi/3) + a_2 \exp(jt2\pi/3) + a_2^* \exp(-jt2\pi/3) \\ &= -a_1 2\cos((t-3)\pi/3) - a_2 \exp(j(t-3)2\pi/3) - a_2^* \exp(-j(t-3)2\pi/3) \\ &= -a_1 2\cos((t-3)\pi/3) - \exp(-j2\pi)(a_2 \exp(jt2\pi/3) + a_2^* \exp(-jt2\pi/3)) \\ &= -a_1 2\cos(t\pi/3 - \pi) - a_2 \exp(jt2\pi/3) - a_2^* \exp(-jt2\pi/3) \\ &= a_1 2\cos(t\pi/3) - a_2 \exp(jt2\pi/3) - a_2^* \exp(-jt2\pi/3) \end{aligned}$$

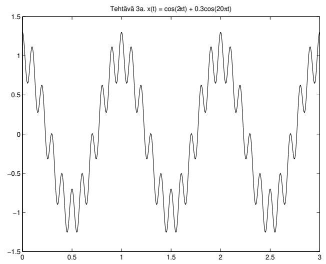
Now, we get $a_2 \exp(jt2\pi/3) + a_2^* \exp(-jt2\pi/3) = 0$ or $a_2 = 0$. This yields $x(t) = a_1 2\cos(t\pi/3) = A\cos(Bt + C)$ where the constants are $A = a_1 2$, $B = \pi/3$ and $C = 0$. Due to the property v) we get $\sum |a_k|^2 = 1/2$ or $2a_1^2 = 1/2$. This yields $a_1 = 1/2$ or $A = 1$. Now, it has been shown that $x(t) = \cos(t\pi/3)$.

24. Filter types: LP / HP / BS / BP.



25. Periodic signal $x(t) = \cos(2\pi t) + 0.3\cos(20\pi t)$.

a) The signal $x(t)$ in time-domain:



b) Let us first determine the basic angular frequency. For the first term $T_1 = 1$ and for the second $T_2 = 0.1$. Therefore, $T_0 = 1$ and $\omega_0 = 2\pi$.

Now, let us present the cosines with complex exponential functions using the basic angular frequency $\omega_0 = 2\pi$:

$$x(t) = a_{-10}e^{j(-10)(2\pi)t} + a_{-1}e^{j(-1)(2\pi)t} + a_1 e^{j(2\pi)t} + a_{10}e^{j(10)(2\pi)t}$$

This gives us the following coefficients:

$$a_{-1} = a_1 = 0.5, \; a_{-10} = a_{10} = 0.15, \; \text{otherwise } a_k = 0$$

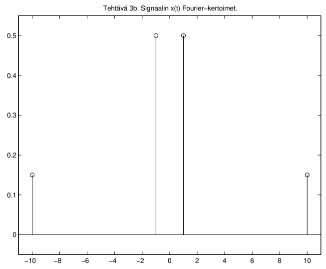


Figure 30: *Fourier-coefficients.*

c) The signal after filtering with an ideal lowpass filter:

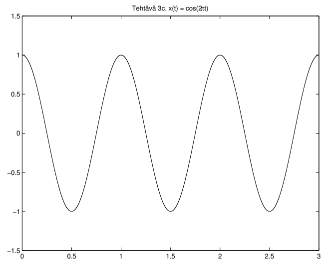


Figure 31: *Lowpass filtered signal.*

d) The signal after filtering with an ideal highpass filter in Figure 32.

26. Consider a mechanical system whose differential equation relating velocity $v(t)$ and the input force $f(t)$ is given by

$$Bv(t) + K\int v(t)dt = f(t).$$

a) The compressive force acting on the spring is $f_s(t) = K\int v(t)dt$. The velocity $v(t)$ can now be written as a function of f_s:

$$v(t) = \frac{1}{K}\frac{d}{dt}f_s(t).$$

Now, the differential equation is as follows:

$$\frac{B}{K}\frac{d}{dt}f_s(t) + f_s(t) = f(t).$$

For an input of the form $f(t) = e^{j\omega t}$, the system yields a response $f_s(t) = H(j\omega)e^{j\omega t}$. Thus,

$$\frac{B}{K}H(j\omega)j\omega e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}.$$

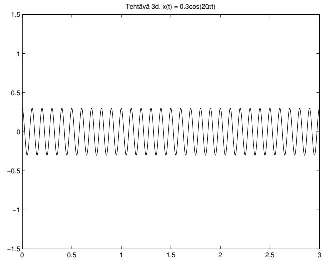


Figure 32: *Highpass filtered signal.*

Let use solve the above equation with regards to the frequency response $H(j\omega)$:

$$H(j\omega) = \frac{1}{1+\frac{B}{K}j\omega} = \frac{1-\frac{B}{K}j\omega}{1+\frac{B^2}{K^2}\omega^2}.$$

Now, the magnitude of the frequency response is

$$|H(j\omega)| = \frac{1}{1+\frac{B^2}{K^2}\omega^2}\sqrt{1+\frac{B^2\omega^2}{K^2}} = \sqrt{\frac{1}{1+\frac{B^2}{K^2}\omega^2}}.$$

The funktion clearly is of the form $f(\omega) \approx \frac{1}{1+\omega}$, i.e, it is a lowpass filter.

b) The compressive force acting on the dashpot is $f_d(t) = Bv(t)$. The velocity $v(t)$ can thus be written as:

$$v(t) = \frac{1}{B}f_d(t)$$

and the differential equation:

$$f_d(t) + \frac{K}{B}\int f_d(t)dt = f(t).$$

Let us solve the equation like in *a)* by substituting $f_d(t) = H(j\omega)e^{j\omega t}$ into the equation. This yields

$$H(j\omega)e^{j\omega t} + \frac{K}{B}\frac{1}{j\omega}H(j\omega)e^{j\omega t} = e^{j\omega t}.$$

This gives us the frequency response:

$$H(j\omega) = \frac{1}{1+\frac{K}{B}\frac{1}{j\omega}} = \frac{1+\frac{K}{B\omega}j}{1+\frac{K^2}{B^2\omega^2}}.$$

By examining the magnitude of the frequency response

$$|H(j\omega)| = \frac{1}{1+\frac{K^2}{B^2\omega^2}}\sqrt{1+\frac{K^2}{B^2\omega^2}} = \sqrt{\frac{1}{1+\frac{K^2}{B^2\omega^2}}},$$

we see that the function is of the form $f(\omega) \approx \frac{1}{1+\frac{\omega}{\omega+1}}$ and thus is a highpass filter.

27. Fourier-transform, sinc-function. Examples 4.4 and 4.5 in the book.

    a) Calculate the Fourier-transform for the signal

$$x(t) = \begin{cases} 1, |t| < T_1 \\ 0, |t| > T_1 \end{cases}$$

$$\begin{aligned} F\{x(t)\} = X(j\omega) &= \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt \\ &= \int_{-T_1}^{T_1} 1\, e^{-j\omega t} dt \\ &= \frac{1}{-j\omega} e^{-j\omega t}\Big|_{-T_1}^{T_1} \\ &= \frac{1}{-j\omega}\left(e^{-j\omega T_1} - e^{-j\omega(-T_1)}\right) \\ &= \frac{1}{j\omega}\left(e^{j\omega T_1} - e^{-j\omega T_1}\right) \\ &= \frac{2}{\omega} sin(\omega T_1), \omega \neq 0 \end{aligned}$$

    When $\omega = 0$, in other words, the DC-component is calculated, the equation above can be expressed

$$X(j\omega) = \frac{2T_1 sin(\omega T_1)}{\omega T_1}$$

    which converges $2T_1$, when $\omega \to 0$, because it is known that $\frac{sin(x)}{x} \to 1$, kun $x \to 0$.

    b) Express the Fourier-transform from a) using sinc-functino, $\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

    $\frac{2}{\omega} sin(\omega T_1) = \frac{2}{\omega}\left(\frac{sin(\pi\frac{\omega T_1}{\pi})}{(\pi\frac{\omega T_1}{\pi})\frac{1}{\omega T_1}}\right) = \frac{2}{\omega}\left(sinc(\frac{\omega T_1}{\pi})(\omega T_1)\right) = 2T_1\, sinc(\frac{\omega T_1}{\pi})$

28. Fourier-transform the following signals and impulse responses using the table (time shift, linearity, derivation in time) and previous results.

    a) $x_1(t) = \begin{cases} 2, & 0 < t < 2 \\ 0, & \text{elsewhere} \end{cases}$

    The signal is the one from problem 1, but delayed by 1 second and multiplied by 2 (draw!). In addition, $T_1 = 1$. From the table it is seen that for the time shifting: if $x(t) \leftrightarrow X(j\omega)$, then $x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$.
    $X_1(j\omega) = 2\, e^{-j\omega 1} X(j\omega) = 2\, e^{-j\omega}\left(2 * 1\, sinc(\frac{\omega 1}{\pi})\right) = 4\, e^{-j\omega}\, sinc(\frac{\omega}{\pi})$

    b) $x_2(t) = \begin{cases} 1, & 0 < t < 1 \\ 3, & 1 < t < 2 \\ 2, & 2 < t < 4 \\ 0, & \text{elsewhere} \end{cases}$

    This can be thought to form as a linear combination of two rectangulars. The first (X_{2a}) is one high in range 0..2 and the other (X_{2b}) two high in range 1..4.
    $X_2(j\omega) = X_{2a}(j\omega) + X_{2b}(j\omega) = e^{-j\omega}(2sinc(\frac{\omega}{\pi})) + 2e^{-j\omega 2.5}(2 * 1.5sinc(\frac{\omega 1.5}{\pi}))$

    c) $h(t) = e^{-(t-2)}u(t-2)$

    Now it is seen that there is a time shifting $t_0 = 2$, and from the table a transform pair is found $e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$, where $a = 1$. So,
    $H_3(j\omega) = e^{-j\omega t_0}\frac{1}{a+j\omega} = e^{-2j\omega}\frac{1}{1+j\omega}$

    d) $x_4(t) = \begin{cases} 1 - |t|, & -1 < t < 1 \\ 0, & \text{elsewhere} \end{cases}$

    In the exercise round 4 in problem 6 there was a similar exercise where the triangle was periodic. Also now the derivation property is used. It is seen (draw!) that
    $\frac{dx_4(t)}{dt} = \begin{cases} 1, & -1 < t < 0 \\ -1, & 0 < t < 1 \end{cases}$
    Now these two regtangulars (advanced $X_{4a'}$, and delayed and inverted $X_{4b'}$):
    $X_{4'}(j\omega) = e^{-j\omega(-0.5)}(2 * 0.5sinc(\frac{\omega 0.5}{\pi})) - e^{-j\omega(0.5)}(2 * 0.5sinc(\frac{\omega 0.5}{\pi}))$
    From the table $\frac{d}{dt}x(t) \leftrightarrow j\omega X(j\omega)$.
    So, $x_{4'}(t) = \frac{dx_4(t)}{dt} \leftrightarrow j\omega X_4(j\omega) = X_{4'}(j\omega)$
    $X_{4'}(j\omega)$ is known, so
    $X_4(j\omega) = \frac{1}{j\omega}X_{4'}(j\omega) = \left(e^{0.5j\omega}(sinc(\frac{\omega}{2\pi})) - e^{-0.5j\omega}(sinc(\frac{\omega}{2\pi}))\right) = \frac{2}{\omega}sinc(\frac{\omega}{2\pi})sin(\frac{\omega}{2}) = \left(sinc(\frac{\omega}{2\pi})\right)^2$
    DC-component can be found to be
    $|X_4(0)| = 1$
    which is the are of rectangular $x_4(t)$.

29. Convolution property (p. 314)

$$y(t) = h(t) * x(t) \leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

    Calculate the convolution $h_1(t) * h_2(t)$ of impulse responses $h_1(t) = e^{-0.5t}u(t)$ and $h_2(t) = 2e^{-t}u(t)$ using convolution property of F-transforms (multiplication of transforms, partial fraction decomposition, inverse transform back to time domain).
    $H_1(j\omega) = \frac{1}{0.5+j\omega}$
    $H_2(j\omega) = 2\,\frac{1}{1+j\omega}$
    $H(j\omega) = H_1(j\omega)\, H_2(j\omega) = \left(\frac{1}{0.5+j\omega}\right)\left(\frac{2}{1+j\omega}\right)$
    Partial fraction decomposition (see basic mathematics or Appendix A)
    $\frac{A}{0.5+j\omega} + \frac{B}{1+j\omega} = \frac{A+Aj\omega+0.5B+Bj\omega}{(0.5+j\omega)(1+j\omega)}$
    $A + 0.5B = 2$ and $A + B = 0$. From these $A = 4, B = -4$.
    $H(j\omega) = \frac{4}{0.5+j\omega} - \frac{4}{1+j\omega}$
    $h(t) = 4e^{-0.5t}u(t) - 4e^{-t}u(t)$
    This property $Y(j\omega) = H(j\omega)X(j\omega)$ is used a lot!

30. Multiplication property (p. 322).

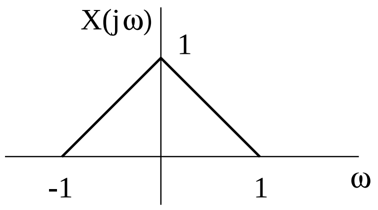


Figure 33: *Fourier-transform of the signal* $x(t)$.

a) Let there be a signal $x(t)$, whose Fourier-transform (spectrum) is $X(j\omega)$ like in the figure 33.

Draw the spectrum $Y(j\omega)$ of signal $y(t) = x(t)p(t)$, when

i) $p(t) = cos(t/2),\ \omega = 0.5,\ T = 4\pi$

ii) $p(t) = cos(t),\ \omega = 1,\ T = 2\pi$

iii) $p(t) = cos(2t),\ \omega = 2,\ T = \pi$

iv) impulse train $p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n\pi),\ T = \pi,\ \omega = 2$, find first F-coefficients of $p(t)$ (p. 299, example 4.8)

This question deals with *amplitude modulation*. Read page 322 in the book.

i) The Fourier series of the function $p(t) = \cos(0.5t)$ is obtained using Euler's equation

$$p(t) = 0.5\left(e^{0.5tj} + e^{-0.5tj}\right),$$

thus $\omega_0 = 0.5$ and $a_{\pm 1} = 0.5$, otherwise a_k is zero. From the tabel it is seen that the frequency of the tranform $k\omega_0$ corresponds $2\pi a_k$. For the cosine there is also a transform pair $cos(\omega_0 t) = \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ Hence, the Fourier transform of the function $y(t)$ is ($\omega_0 = 0.5$)

$$Y(j\omega) = 0.5X\left(j(\omega + 0.5)\right) + 0.5X\left(j(\omega - 0.5)\right).$$

The Fourier spectrum is drawn in the figure 34 below.

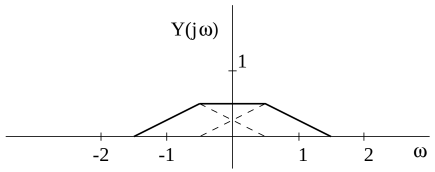


Figure 34: $\omega_0 = 0.5$

ii) This function resembles the function before, but $\omega_0 = 1$ and $a_{\pm 1} = 0.5$, the other a_k's are zero. The Fourier transform is ($\omega_0 = 1$)

$$Y(j\omega) = 0.5X\left(j(\omega + 1)\right) + 0.5X\left(j(\omega - 1)\right),$$

and the Fourier spectrum looks like (figure 35)

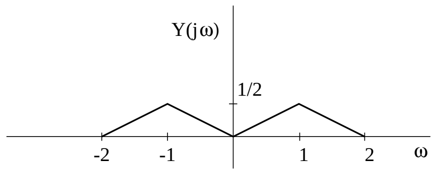


Figure 35: $\omega_0 = 1$

iii) Now we have $\omega_0 = 2$ and $a_{\pm 1} = 0.5$. The Fourier transform is ($\omega_0 = 2$)

$$Y(j\omega) = 0.5X\left(j(\omega + 2)\right) + 0.5X\left(j(\omega - 2)\right),$$

and the Fourier spectrum looks like (figure 36)

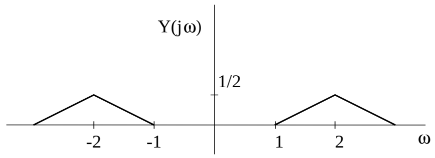


Figure 36: $\omega_0 = 2$

iv) Clearly, this function is periodic, with fundamental period $T = \pi$. Let us calculate the Fourier co-efficients of the function $p(t)$:

$$a_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \sum_{n=-\infty}^{\infty} \delta(t - n\pi)e^{jk2t}dt.$$

The delta function is zero, when $t = n\pi$. This is satisfied, at the period investigated, when $n = 0$, and thus $t = 0$. Hence, the Fourier co-efficients of $p(t)$ are $a_k = \frac{1}{\pi}$ for all k, and the fundamental frequency is $\omega_0 = 2\pi/T = 2$. Thus, the Fourier transform is

$$\sum_{-\infty}^{\infty} \frac{1}{\pi} X\left(j(\omega - 2n)\right),$$

and the spectrum is as drawn below (figure 37).

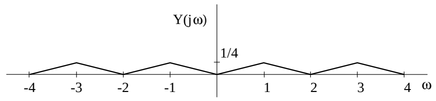


Figure 37: Impulse train

b) Calculate the F-transform for the signals $x(t) = \left(\frac{sin(\pi t)}{\pi t}\right)\left(\frac{sin(2\pi(t-1))}{\pi(t-1)}\right)$
Again the transform with convolution. The table from page 329

$$X_1(j\omega) = \begin{cases} 1, |\omega| < \pi \\ 0, |\omega| > \pi \end{cases} .$$

Notice that X_2 is similar, but shifted in time, so

$$X_2(j\omega) = \begin{cases} e^{-j\omega}, |\omega| < 2\pi \\ 0, |\omega| > 2\pi \end{cases} .$$

Calculate next the convolution which can be done in three parts

$$X(j\omega) = X_1(j\omega) * X_2(j\omega) = \begin{cases} \int_{-2\pi}^{w+\pi} e^{-j\omega} d\omega, & -3\pi \le w \le -\pi \\ \int_{w-\pi}^{w+\pi} e^{-j\omega} d\omega, & -\pi \le w \le \pi \\ \int_{w-\pi}^{2\pi} e^{-j\omega} d\omega, & \pi \le w \le 3\pi \end{cases} ,$$

in other words, in the first parteli X_1 and X_2 are covered partly, in the second totally and in the third again partly. After integration and substitution

$$X(j\omega) = \begin{cases} -\frac{1}{j}(e^{-j(w+\pi)} - e^{2\pi}) \\ -\frac{1}{j}(e^{-j(w+\pi)} - e^{-j(w-\pi)}) \\ -\frac{1}{j}(e^{-2\pi j} - e^{-j(w-\pi)}) \end{cases} = \begin{cases} \frac{1}{j}(1 + e^{jw}), & -3\pi \le w \le -\pi \\ 0, & -\pi \le w \le \pi \\ -\frac{1}{j}(1 + e^{-jw}), & \pi \le w \le 3\pi \end{cases} .$$

31. Calculate the Fourier transforms of the following signals:

a) $x[n] = (\frac{1}{2})^{n-1} u[n-1]$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (\frac{1}{2})^{n-1} u[n-1] e^{-j\omega n} \\ &= \sum_{n=1}^{\infty} (\frac{1}{2})^{n-1} e^{-j\omega n} = 2\sum_{n=1}^{\infty} (\frac{1}{2})^{n} e^{-j\omega n} \\ &= 2 \cdot \frac{1}{2} e^{-j\omega} \sum_{n=0}^{\infty} (\frac{1}{2})^{n} e^{-j\omega n} = \frac{e^{-j\omega}}{1 - \frac{1}{2} e^{-j\omega}} \end{aligned}$$

b) $x[n] = \delta[n-1] + \delta[n+1]$

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (\delta[n-1] + \delta[n+1]) e^{-j\omega n} \\ &= e^{-j\omega} + e^{j\omega} = 2\cos\omega \end{aligned}$$

32. Given that $X(e^{j\omega})$ is the Fourier transform of signal $x[n]$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$ (see Table 5.1 p. 391 in the course book).

a) $x_1[n] = x[1-n] + x[-1-n] = x[-n+1] + x[-n-1]$

Here we have a sum of two signals, where $x[n]$ is first time-reversed and then advanced/delayed with one time step.

$$X_1(e^{j\omega}) = e^{j\omega} X(e^{-j\omega}) + e^{-j\omega} X(e^{-j\omega}) = 2\cos\omega X(e^{-j\omega})$$

b) $x_2[n] = \frac{1}{2}(x^*[-n] + x[n])$

The required Fourier transform of $x^*[n]$ is $X^*(e^{-j\omega})$:

$$X_2(e^{j\omega}) = \frac{1}{2}(X^*(e^{j\omega}) + X(e^{j\omega})) = \frac{1}{2}(2\,\Re e\{X(e^{j\omega})\}) = \Re e\{X(e^{j\omega})\}$$

c) $x_3[n] = (n-1)^2 x[n] = n^2 x[n] - 2nx[n] + x[n]$

Multiplication by n in the time-domain corresponds to derivation in the frequency-domain (with the coefficient j):

$$\begin{aligned} X_3(e^{j\omega}) &= j^2 \frac{d^2}{d\omega^2} X(e^{j\omega}) - 2j \frac{d}{d\omega} X(e^{j\omega}) + X(e^{j\omega}) \\ &= -\frac{d^2}{d\omega^2} X(e^{j\omega}) - 2j \frac{d}{d\omega} X(e^{j\omega}) + X(e^{j\omega}) \end{aligned}$$

33. a) From the definition of the discrete Fourier transform, we see that

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j \cdot 0n} = \sum_{n=-\infty}^{\infty} x[n] = 6.$$

b) By examining the signal, we see that the time-shift $x[n+2]$ produces a real and even signal $r[n]$, whose Fourier transform is known also to be real and even (arg $R(e^{j\omega}) = 0$). The same property also holds in reverse, so by constructing the above real and even signal and then delaying it by two time-steps, $r[n-2]$, we get the original signal $x[n]$. In the frequency domain, this means that the frequency response of the original signal is obtained by a Fourier transform $R(e^{j\omega})$ of a real signal which is then time-shifted with $e^{-j\omega n_0}$:

$$X(e^{j\omega}) = e^{-2j\omega} R(e^{j\omega}) ,$$

where the coefficient of $R(e^{j\omega})$ results from the time-shift ($n_0 = 2$). The above equation of $X(e^{j\omega})$ is now in the form $X(e^{j\omega}) = e^{j \arg X(e^{j\omega})} |X(e^{j\omega})|$, so the angle is obtained directly: $\arg X(e^{j\omega}) = -2\omega$.

The angle can also be computed as follows:

$$X(j\omega) = e^{-2j\omega} R(j\omega) = (\cos 2\omega - j \sin 2\omega) R(j\omega),$$

$$\arg X(j\omega) = \tan^{-1}\left[\frac{-R(j\omega)\sin 2\omega}{R(j\omega)\cos 2\omega}\right] = -2\omega.$$

c) From the definition of the inverse discrete Fourier transform, we see that

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

Now, we notice that

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega \Leftrightarrow \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega = 2\pi x[0] \Leftrightarrow \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) d\omega = 4\pi,$$

when $x[0] = 2$.

d) Using the equation of the discrete Fourier transform:

$$X\left(e^{j\pi}\right) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\pi n}.$$

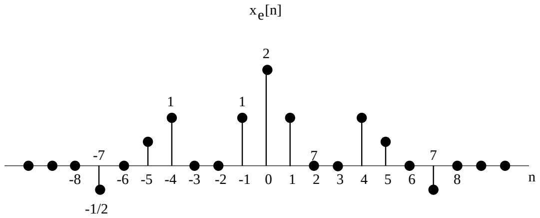
When n is even, $e^{-j\pi n} = 1$ whereas when n is odd, $e^{-j\pi n} = -1$. Therefore, we get

$$X\left(e^{j\pi}\right) = \sum_{n=-\infty}^{\infty} x[n](-1)^n = 1 - 1 + 2 - 1 - 1 + 2 - 1 + 1 = 2.$$

e) By examining the Table 5.1 on page 391, we see that

$$\Re e\{X\left(e^{j\omega}\right)\} \longleftrightarrow x_e[n] = \frac{1}{2}\{x[-n] + x[n]\}.$$

Therefore, the even signal is as presented below.



f) By using the Parseval relation,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega = \sum_{n=-\infty}^{\infty} |x[n]|^2,$$

or

$$\int_{-\pi}^{\pi} \left|X\left(e^{j\omega}\right)\right|^2 d\omega = 2\pi \cdot (1 + 1 + 4 + 1 + 1 + 4 + 1 + 1) = 28\pi.$$

For the derivation in the frequency-domain, it holds that:

$$j\frac{d}{d\omega}X(e^{j\omega}) \leftrightarrow nx[n].$$

With the Parseval relation, we can now write:

$$\int_{-\pi}^{+\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} |-jnx[n]|^2 = 2\pi \sum_{n=-\infty}^{\infty} n^2 x[n]^2.$$

And the results is

$$\int_{-\pi}^{+\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega = 2\pi(9 + 1 + 1 + 9 + 64 + 25 + 49) = 316\pi.$$

34. 1) $\Re e\{X(e^{j\omega})\} = 0$, if $x[n]$ is real and odd. Signals d) and f) are odd.

2) $\Im m\{X(e^{j\omega})\} = 0$, if $x[n]$ is real and even. The only even signal is b).

3) $e^{j\alpha\omega}X(e^{j\omega})$ is real, if $x[n]$ can be made real and even with some time-shift α. First, all signals are real. Second, the evenness can be obtained for b) (which is already even, i.e. with $\alpha = 0$), for d) (by shifting the signal odd α steps either to the right or to the left, i.e. with $\alpha = 2k+1,\ k \in \mathbb{Z}$), and for e) ($\alpha = -1$).

4) By using the definition of the Fourier transform, we get (as in exercise 3c):

$$2\pi x[0] = \int_{-\pi}^{\pi} X\left(e^{j\omega}\right).$$

$x[0] = 0$ for signals c), d), e), and f).

5) $X(e^{j\omega})$ is periodic for all discrete-time Fourier transforms.

6) Like in exercise 3a, we get:

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 0.$$

This condition is fulfilled with signals d) and f).

The results of the exercise gathered into a table:

	1)	2)	3)	4)	5)	6)
a)	false	false	false	false	true	false
b)	false	true	true	false	true	false
c)	false	false	false	true	true	false
d)	true	false	true	true	true	true
e)	false	false	true	true	true	false
f)	true	false	false	true	true	true

35. The equation presented in the exercise represents the periodic convolution of X and G (see pages 388–389 in the book). After a minor modification, the convolution can written using the convolution operator as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = X'(e^{j\omega}) * G(e^{j\omega}) = 1 + e^{-j\omega}$$

Now, due to the convolution property, the Fourier transform of $y[n] = x[n]g[n]$ is $1 + e^{-j\omega}$. The signal having a Fourier transform of this form is, according to the Table 5.2,

$$y[n] = F^{-1}\{1 + e^{-j\omega}\} = \delta[n] + \delta[n-1] = \begin{cases} 1, & n = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

The solution can be obtained, for example, graphically, by sketching $x[n]$, the desired $g[n]$, and $y[n]$ and by seeking a fitting $g[n]$. In a), we get $g[n] = y[n]/x[n] = 1/(-1)^n, n = 0, 1$ or

$$g[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

In b), we get $g[n] = y[n]/x[n] = 1/(1/2)^n, n = 0, 1$ or

$$g[n] = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

36. When "sketcing" frequency response it is smart to calculate a table of $H(e^{j\omega})$ with some "important" frequencies like $\omega = \{0, \pi/2, \pi, ...\}$.

Let there be frequency response $H(e^{j\omega}) = 1 + e^{-j\omega}$. Frequency Response can be decomposed to amplitude and phase responses $H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\ arg\{H(e^{j\omega})\}}$.

a) *Sketch the curve of* $H(e^{j\omega})$ *in complex plane, when* ω *gets values* $0..\pi$.
Calculate values for $H(e^{j\omega})$, when $\omega = 0, \pi/4, \pi/2, 3\,\pi/4, \pi$. Interpolate $1 + e^{j\omega}$ and sketch the figure 38.
With Matlab:

```
w = [ 0 :  pi/256 :  pi];
H = 1 + exp(-j*w);
plot(w, H)
```

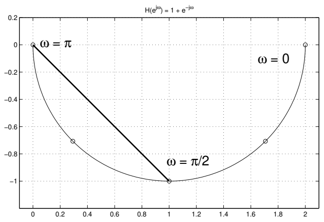


Figure 38: $H(e^{j\omega})$ *in complex plane. Frequency as a parameter in curve (not x- or y-axis!). The vector from origo shows the situation* $\omega = \pi/2$, *when* $H(e^{j\omega}) = 1 - j$, *that is,* $|H(e^{j\omega})| = \sqrt{2}$ *and arg* $\{H(e^{j\omega})\} = -\pi/4$.

The amplitude of frequency response with required frequency range can be received as origovector length from origo to the point in the curve where the vector points. The phase of frequency response with required range can be found to be the angle of the vector and x-axis.

b) *Calculate amplitude response* $|H(e^{j\omega})|$.
The absolute value of a complex number, kuva 39 can be got:

$$\begin{aligned} |H(e^{j\omega})| &= |1 + e^{-j\omega}| = |(1 + cos(\omega)) + j\,(-sin(\omega))| \\ &= \sqrt{(1 + 2\,cos(\omega) + cos^2(\omega)) + sin^2(\omega)} = \sqrt{2 + 2\,cos(\omega)} \end{aligned}$$

or

$$\begin{aligned} |H(e^{j\omega})| &= \sqrt{H(e^{j\omega})H^*(e^{j\omega})} = \sqrt{H(e^{j\omega})H(e^{-j\omega})} \\ &= \sqrt{(1 + e^{-j\omega})(1 + e^{j\omega})} = \sqrt{1 + e^{j\omega} + e^{-j\omega} + 1} = \sqrt{2 + 2\,cos(\omega)} \end{aligned}$$

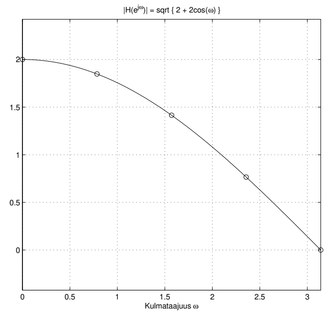


Figure 39: $|H(e^{j\omega})|$

The plotting of amplitude response in Matlab:
`plot(w,abs(H));`

c) Calculate first phase response $arg\{H(e^{j\omega})\}$, kuva 40, using the exponential function or using the table (trigonometric angles):

$$\begin{aligned} arg\{H(e^{j\omega})\} &= arg\{1 + e^{-j\omega}\} \\ &= arg\{(e^{-0.5j\omega})(e^{0.5j\omega} + e^{-0.5j\omega})\} \\ &= arg\{(e^{-0.5j\omega})\} + arg\{(e^{0.5j\omega} + e^{-0.5j\omega})\} \\ &= arg\{(e^{-0.5j\omega})\} + 0 = -0.5\omega \end{aligned}$$

or equally

$$\begin{aligned} arg\{H(e^{j\omega})\} &= arg\{1 + e^{-j\omega}\} = arg\{(1 + cos(\omega)) + j\,(-sin(\omega))\} \\ &= arctan\{\frac{-sin(2\frac{\omega}{2})}{1 + cos(2\frac{\omega}{2})}\} = arctan\{\frac{-2sin(\frac{\omega}{2})cos(\frac{\omega}{2})}{1 + 2cos^2(\frac{\omega}{2}) - 1}\} = arctan\{\frac{-sin(\frac{\omega}{2})}{cos(\frac{\omega}{2})}\} = - \end{aligned}$$

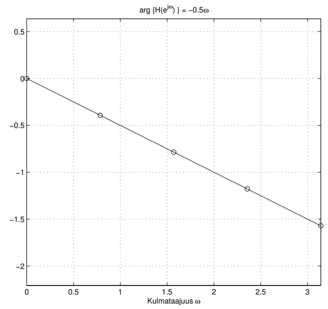


Figure 40: $arg\{H(e^{j\omega})\}$

The plotting of phase response in Matlab:
`plot(w,angle(H));`

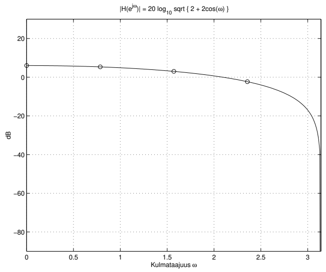


Figure 41: $20\,log_{10}|H(e^{j\omega})|$

d) Decibel-scale, compare figures 39 and 41. In the latter one $20\,log_{10}|H(e^{j\omega})|$. Notice that $20\,log_{10}\,1 = 0\;dB \doteq 1$. For example, $20\,log_{10}\,2 = 6.02\;dB$, kun $\omega = 0$.
The plotting of amplitude response in Matlab:
`plot(w,20*log10(abs(H)));`
or both amplitudi and phase response
`freqz([1 1],1);`,
where [1 1] and 1 are the coefficients of numerator and denumerator of $H(e^{j\omega})$, using descending power of $e^{-j\omega}$ polynomials.

e) Group delay $\tau(\omega) = -\frac{d}{d\omega}arg\{H(e^{j\omega})\}$.

$$\tau(\omega) \;=\; -\frac{d}{d\omega}arg\{H(e^{j\omega})\} = \frac{d}{d\omega}(-0.5\omega) = 0.5$$

37. The Matlab command `freqz` plots frequency range $0..\pi$, because **the Fourier-transform of discrete-time real signal is periodic** in a way that amplitude response is even and phase response odd with the period of 2π. In other words, if $H(e^{j\omega})$ is known in range $0..\pi$, then for $-\pi..0$ it holds that $|H(e^{j\omega})| = |H(e^{-j\omega})|$ and $\angle H(e^{-j\omega}) = -\angle H(e^{j\omega})$.

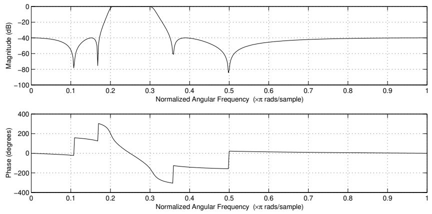


Figure 42: $H(e^{j\omega})$: amplitude response and phase response

a) This is an elliptic IIR bandpassfilter of 8th degree. From the figure 42 it is obvious to plot figures 43 and 44. Also the frequency scale was different.

b) As mentioned before, Fourier-transform for discrete-time sequences (DTFT) is always periodic. In the end of the course there is a concept of sampling theorem, where

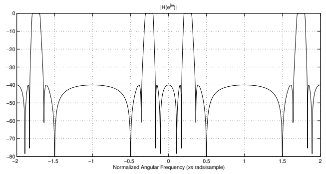


Figure 43: $|H(e^{j\omega})|$ *when* $-2\pi..2\pi$

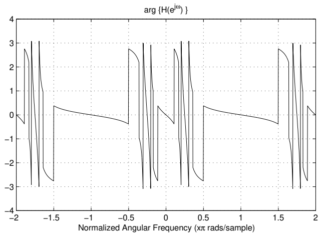


Figure 44: $arg\{H(e^{j\omega})\}$ *when* $-2\pi..2\pi$

the main idea is that in DTFT can be observed only those frequencies who are less than half of the sampling frequency. The observation points goes on the unit circle $(e^{j\omega})$ and the period is therefor 2π.
From the definition $H(e^{j\omega}) = \sum h[n]\,e^{-j\omega n}$.
When $\omega_1 = \frac{\pi}{2}$ ja $\omega_2 = \frac{5\pi}{2}$, and n is integer, then

$$\begin{aligned} H(e^{j5\frac{\pi}{2}}) &= \sum h[n]\,e^{-j5\frac{\pi}{2}n} = \sum h[n]\,e^{-j\frac{\pi}{2}n}e^{-j4\frac{\pi}{2}n} \\ &= \sum h[n]\,e^{-j\frac{\pi}{2}n} \\ &= H(e^{j\frac{\pi}{2}}) \end{aligned}$$

c) The Fourier-transform of continous-time signals is not periodic. It is possible to observe smaller and smaller time periods, in other words, higher and higher frequencies.

38. The convolution in time-domain corresponds the multiplication of transforms in frequency-domain $y[n] = h[n] * x[n] \leftrightarrow Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$. The LTI-system S is characterized by its impulse function $h[n] = \frac{1}{3}(-\delta[n] + 2\,\delta[n-1] - \delta[n-2])$.

a) The block diagram S is in the figure 45.

b) The result is obtained either using transform formulae $h[n] \leftrightarrow H(e^{j\omega})$, $\delta[n] \leftrightarrow 1$, $\delta[n-n_0] \leftrightarrow e^{jn_0\omega}$ or by calculating Fourier-transform by definition:

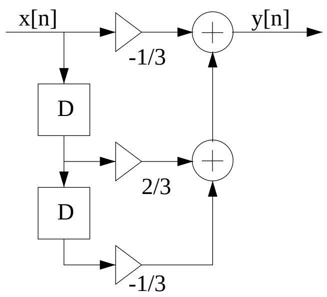


Figure 45: *Problem 38(a) Block diagram*

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=0}^{2} h[n]e^{-j\omega n} \\
&= \frac{1}{3}(-1+2\,e^{-j\omega}-e^{-2j\omega}) = e^{-j\omega}\left(-\frac{1}{3}(e^{j\omega}-2+e^{-j\omega})\right) = e^{-j\omega}\left(\frac{2\,(1-cos(\omega))}{3}\right)
\end{aligned}
$$

So, $|H(e^{j\omega})| = \frac{2\,(1-cos(\omega))}{3}$ and arg $\{H(e^{j\omega})\} = -\omega$.

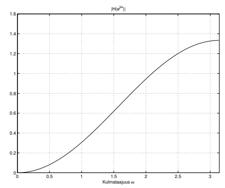


Figure 46: *Problem 38(b) Amplitude Response*

Amplitude Response $|H(e^{j\omega})|$ (figure 46 gets value 0, when frequency $\omega = 0$ and maximum value 1, when $\omega = \pi$ growing monotonically. So the filter is lowpass, which amplifies changes in signal and attenuates constant signal.

c) Response $y[n] = x[n] * h[n]$ or $y[n] = F^{-1}\{F\{x[n]\}\ F\{h[n]\}\}$. In the figure 47 the response $y[n]$ for inputs (A=1)

i) constant sequence, amplitude A, $\{A, A, A, ...\}$

ii) sequence $\{A, A, A, A, A, A, -A, -A, -A, -A, -A, -A, ...\}$

iii) periodic sequence $\{A, -A, A, -A, ...\}$

d) The LTI-system was drawn in a). We can see that it is causal. The system is stable because it is FIR (finite impulse response): bounded input produces bounded output. (The other type is IIR, infinite impulse response, which can be unstable, if feedback is too strong.)

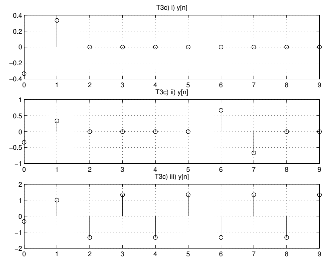


Figure 47: *Problem 38(c) $y[n]$*

39. The module $|H(j\omega)|$ is the length of the complex number $H(j\omega) = |H(j\omega)|e^{j\arg H(j\omega)}$. by manipulating the frequency response

$$
\begin{aligned}
H(j\omega) &= \frac{1-j\omega}{1+j\omega} \\
&= \frac{\sqrt{1+\omega^2}e^{-j\alpha}}{\sqrt{1+\omega^2}e^{j\alpha}} \\
&= \frac{e^{-j\alpha}}{e^{j\alpha}} \\
&= e^{-j2\alpha}
\end{aligned}
$$

where $\alpha = \arctan\omega$.

Then $|H(j\omega)| = 1 = A$, so this is so called all-pass-filter, whose amplification is constant for all frequencies.

The group delay $\tau(\omega) = -\frac{d}{d\omega}(-2\arctan(\omega)) = -\frac{d}{d\omega}(2\arctan(\omega)) = \frac{2}{1+\omega^2}$. In this case, the group delay is always positive.

40. The exponential function $x(t) = e^{j\omega t}$ is the eigenfunction of a LTI-system. Since

$$
\begin{aligned}
x(t) = \cos(\omega_0 t + \phi_0) &= \frac{1}{2}\left[e^{j(\omega_0 t+\phi_0)} + e^{-j(\omega_0 t+\phi_0)}\right] \\
&= \frac{1}{2}e^{j\phi_0}e^{j\omega_0 t} + \frac{1}{2}e^{-j\phi_0}e^{-j\omega_0 t}
\end{aligned}
$$

it follows that

$$
y(t) = \frac{B}{2}e^{j\phi_0}e^{j\omega_0 t} + \frac{C}{2}e^{-j\phi_0}e^{-j\omega_0 t}
$$

where $B = H(j\omega_0)$ and $C = H(-j\omega_0)$. Sincea $h(t)$ is real then $C = B^*$ or $B = |H(j\omega_0)|e^{j\arg H(j\omega_0)}$ and $C = |H(j\omega_0)|e^{-j\arg H(j\omega_0)}$. If follows

$$
\begin{aligned}
y(t) &= \frac{1}{2}|H(j\omega_0)|\left[e^{j\arg H(j\omega_0)}e^{j(\omega_0 t+\phi_0)} + e^{-j\arg H(j\omega_0)}e^{-j(\omega_0 t+\phi_0)}\right] \\
&= \frac{1}{2}|H(j\omega_0)|\left[e^{j(\omega_0 t+\phi_0+\arg H(j\omega_0))} + e^{-j(\omega_0 t+\phi_0+\arg H(j\omega_0))}\right] \\
&= |H(j\omega_0)|\cos(\omega_0 t+\phi_0+\arg H(j\omega_0))
\end{aligned}
$$

This can be written

$$y(t) = |H(j\omega_0)|\cos(\omega_0(t + \frac{\arg H(j\omega_0)}{\omega_0}) + \phi_0) = |H(j\omega_0)|x(t + \frac{\arg H(j\omega_0)}{\omega_0})$$

And, finally $A = |H(j\omega_0)|$ and $t_0 = -\frac{\arg H(j\omega_0)}{\omega_0}$.

The result would have been obtained even more easily, using $y(t) = Ax(t - t_0)$. $Y(j\omega) = AX(j\omega)e^{j\omega_0 t_0}$ and $H(j\omega) = Y(j\omega)/X(j\omega) = Ae^{j\omega_0 t_0}$. From this it follows that $A = |H(j\omega_0)|$. In addition $\omega_0 t_0 = \arg H(j\omega)$ or $t_0 = \frac{\arg H(j\omega)}{\omega}$.

41. Sketch amplitude/phase responses of $x[n] = cos(0.2\pi n) + 2\,cos(0.05\pi n) + 0.1\,\epsilon[n]$, when $\epsilon[n]$ is gaussian white noise. MATLAB: `help randn`.

42. Linear and nonlinear phase. SKIP!

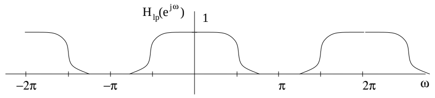
43. a) Let us modify the impulse response as follows:

$$h_{hp}[n] = (-1)^n h_{lp}[n] = e^{-j\pi n}h_{lp}[n].$$

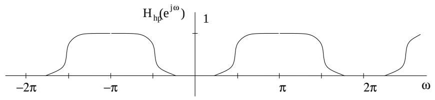
The Fourier transform is now

$$H_{hp}\left(e^{j\omega}\right) = H_{lp}\left(e^{j(\omega+\pi)}\right),$$

i.e., the frequency response is shifted forward by the amount of π. Let us sketch the frequency response of a lowpass filter:



and the frequency response of the corresponding H_{hp}:



$\Rightarrow H_{hp}$ is clearly a frequency response of a highpass filter.

b) Let us proceed inversely to part a):

$$h_{lp}[n] = (-1)^n h_{hp}[n] = e^{-j\pi n}h_{hp}[n].$$

The Fourier transform is now

$$H_{lp}\left(e^{j\omega}\right) = H_{hp}\left(e^{j(\omega+\pi)}\right).$$

44. a) The Fourier transform of the system is

$$j\omega Y(j\omega) + 2Y(j\omega) = X(j\omega),$$

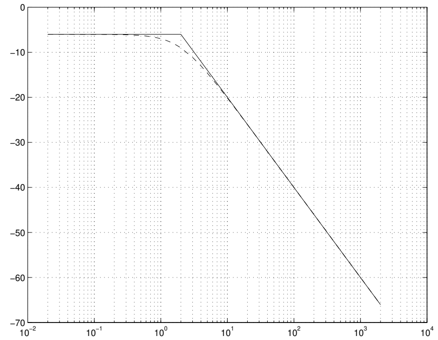
which yields the frequency response

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{2+j\omega} = \frac{2-j\omega}{4+\omega^2}.$$

For the Bode plot, let us calculate the amplitude of the frequency response is decibels:

$$20\log_{10}|H(j\omega)| = 20\log_{10}\sqrt{\frac{1}{4+\omega^2}} = -10\,\lg(4+\omega^2).$$

The -3dB point of the amplitude plot, i.e. the point where $\frac{1}{4+\omega^2}$ is one half of its maximum value, is $\omega = 2$. This enables us to sketch the Bode diagram. In the asymptotic approximation of the amplitude, the amplitude remains constant until the 3dB point. After that, the amplitude decreases 20dB/decade. The approximation is as follows:



The actual amplitude response is shown with the dashed line.

The phase response is as follows:

$$\arg H(j\omega) = \arctan(-\omega/2) = -\arctan(\omega/2).$$

The asymptotic approximation of the phase is obtained by assuming that the phase decreases by the amount of $\pi/2$ from its initial value in the range $[0.1\omega_{3dB}, 10\omega_{3dB}]$, i.e. in the decades surrounding the 3dB point. The approximation and the actual phase response are as follows:

b) The group delay:

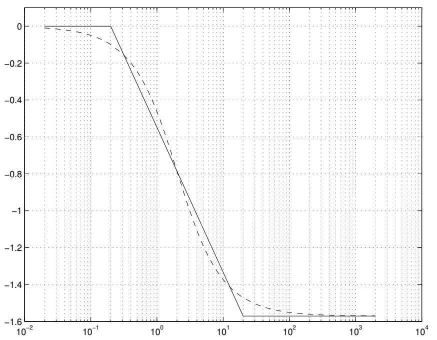
$$\tau(\omega) = -\frac{d}{d\omega}\{\arg H(j\omega)\} = -\frac{d}{d\omega}\{-\arctan(\omega/2)\} = \frac{1}{2}\frac{1}{1+(\omega/2)^2} = \frac{2}{4+\omega^2}.$$

c) The Fourier transform of the input is

$$X(j\omega) = \frac{1}{1+j\omega}.$$

Thus, the Fourier transform of the output is

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{2+j\omega}\cdot\frac{1}{1+j\omega}.$$



d) Now, let us use the partial fraction decomposition on the above Fourier transform.

$$\frac{A}{2+j\omega}+\frac{B}{1+j\omega}=\frac{1}{2+j\omega}\cdot\frac{1}{1+j\omega}.$$

By multiplying with the denominator of the right-hand side, we get

$$A(1+j\omega)+B(2+j\omega)=1,$$

where we get a simple system of equations

$$\begin{cases} A+2B=1 \\ A+B=0 \end{cases} = \begin{cases} -B+2B=1 \\ A=-B \end{cases}$$

Therefore, we get the values of the coefficients as $A=-1$ and $B=1$. The partial fraction decomposition is now

$$\frac{1}{1+j\omega}-\frac{1}{2+j\omega}$$

and thus the output in the time-domain

$$y(t)=e^{-t}u(t)-e^{-2t}u(t).$$

45. a) $H_{lp}(j\omega)$ has the same frequency response as H_0 except the scaling with the constant a. Therefore the new cut-off frequency is

$$\omega_c=a\omega_0 \Leftrightarrow a=\frac{\omega_c}{\omega_0}.$$

b)

$$H_{lp}(j\omega)=H_0(j\frac{\omega_c}{\omega_0}\omega).$$

By using the tables, we get

$$h_{lp}=\frac{\omega_0}{\omega_c}h_0\left(\frac{\omega_0}{\omega_c}t\right).$$

c) As above, we get

$$s_{lp}=\frac{\omega_0}{\omega_c}s_0\left(\frac{\omega_0}{\omega_c}t\right).$$

d) From part c), we see that the step response is also scaled with the constant $1/a$. The rise time is thus

$$\tau_{lp}=\frac{\omega_0}{\omega_c}\tau_0.$$

It seems that here we have a trade-off: by decreasing the cut-off frequency we increase the rise time of the filter.

46. (a) The frequency response can be calculated with the equation

$$Y(\exp(j\omega))=H(\exp(j\omega))X(\exp(j\omega))$$

Let us calculate the Fourier transforms of both sides of the difference equation:

$$Y(\exp(j\omega))-\frac{1}{6}\exp(-j\omega)Y(\exp(j\omega))-\frac{1}{6}\exp(-j2\omega)Y(\exp(j\omega))=X(\exp(j\omega))$$

By regrouping the terms and substituting $H=Y/X$, we get

$$H(\exp(j\omega))=\frac{1}{(1-\frac{1}{2}\exp(-j\omega))(1+\frac{1}{3}\exp(-j\omega))}$$

(b) Let us write the frequency response as

$$H(\exp(j\omega))=\frac{A}{1-\frac{1}{2}\exp(-j\omega)}+\frac{B}{1+\frac{1}{3}\exp(-j\omega)}$$

First, let us solve $1-\frac{1}{2}\exp(-j\omega)=0$. This is satisfied by an ω with $\exp(-j\omega)=2$ or $\omega=j\log 2$. Now, $(1-\frac{1}{2}\exp(-j\omega))H(\exp(j\omega))=A=\frac{1}{1+\frac{2}{3}}=\frac{3}{5}$. Let us solve B by substituting ω with an arbitrary constant. As the constant, let us use $\omega=0$. Then, $H(0)=\frac{1}{(1-\frac{1}{2})(1+\frac{1}{3})}=\frac{1}{\frac{2}{3}}=\frac{3}{2}$. On the other hand, $H(0)=2A+\frac{3}{4}B=\frac{6}{5}+\frac{3}{4}B$. Now, we get $B=\frac{4}{3}(\frac{3}{2}-\frac{6}{5})=2-\frac{8}{5}=\frac{2}{5}$. As the inverse transform of $\frac{1}{1-a\exp(-j\omega)}$ is $a^n u[n]$, we get

$$h([n]=\frac{3}{5}(\frac{1}{2})^n u[n]+\frac{2}{5}(-\frac{1}{3})^n u[n]$$

47. (a) The Fourier transform of the signal $f(t)=\frac{\sin\omega_c t}{\pi t}$ is

$$F(j\omega)=\begin{cases} 1, & |\omega|<\omega_c \\ 0, & |\omega|>\omega_c \end{cases}$$

Now, $H(j\omega)=\frac{1}{2\pi}F(j\omega)*G(j\omega)$ i.e. convolution. By selecting $G(j\omega)=2\pi\delta(\omega-2\omega_c)+2\pi\delta(\omega+2\omega_c)$, we get the desired result since the convolution with a unit impulse results in a time-shift of the original signal. The signal $g(t)$ is obtained by an inverse transform using the Table 4.2 on page 329. Therefore, we get $g(t)=2\cos(2\omega_c t)$.

(b) The impulse response is gathered towards the origin.

48. Terminology

a) **sampling process**, converting continous-time signal into discrete by sampling, processing/filtering, and reconstruction back to continouos. See page 534 or slid 11 from Lec-7.

b) **impulse train**, see the problem 2.

c) **prefiltering**, sampling theorem insists bandlimited signal where prefiltering aims to.

d) **signal reconstruction**, from samples to continuos signal, requires interpolation

e) **zero order hold**, simple interpolation

f) **aliasing**, phenomen if requirements for sampling theorem are not met.

49. Impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Because $p(t)$ is periodic with period T, it can be express as Fourier-series.

$$p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j(2\pi nt/T)}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-j(2\pi nt/T)} dt = \frac{1}{T} e^{-j(2\pi nt/T)}|_{t=0} = \frac{1}{T}$$

for each n, because the definition of integral of $\delta(t)$. Thus

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi nt/T)}$$

50. Sampling and aliasing. A better version: Problem 23 in Extra Material T-61.140 from the web page.

a) The simpliest might be $\theta = -\pi/4 = 2\pi\frac{-1}{8}$, which is going clockwise. But the same result is obtained, if going counterwise $\theta = 7\pi/4 = 2\pi\frac{7}{8} = 2\pi\frac{-1}{8} + 2\pi$. This is used in b).

In general $\theta = -\pi/4 + 2\pi k$.

b) There is a continuous-time signal

$$x(t) = sin(2\pi f_1 t) + sin(2\pi f_2 t) - sin(2\pi f_3 t)$$

Let $f_1 = 100Hz$, $f_2 = 300Hz$ and $f_3 = 700Hz$. (It is also possible to use $f = 2\pi\omega$, if angle frequency is needed.) Solution to this exercise begins from page 3.

Signal in frequency domain.

Because $x(t)$ is periodic with basic frequency 100 Hz, it is possible to derive Fourier series $X(j\omega)$. If $f_1 = f_0$, then $f_2 = 3f_0$ ja $f_3 = 7f_0$, and using Fourier-series

$$x(t) = \sum_{k=-\infty}^{k=\infty} a_k e^{jk(2\pi f_0)t}$$

and exponent function

$$x(t) = \frac{1}{2j}\left((e^{j2\pi f_1 t} - e^{-j2\pi f_1 t}) + (e^{j2\pi f_2 t} - e^{-j2\pi f_2 t}) - (e^{j2\pi f_3 t} - e^{-j2\pi f_3 t})\right)$$

$$= \frac{1}{2j}\left((e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) + (e^{j2*3\pi f_0 t} - e^{-j2*3\pi f_0 t}) - (e^{j2*7\pi f_0 t} - e^{-j2*7\pi f_0 t})\right)$$

we get Fourier series coefficients

$$a_{-7} = \frac{1}{2j} = \frac{1}{2}(-j)$$

$$a_{-3} = -\frac{1}{2j} = \frac{1}{2}j$$

$$a_{-1} = -\frac{1}{2j} = \frac{1}{2}j$$

$$a_{1} = \frac{1}{2j} = \frac{1}{2}(-j)$$

$$a_{3} = \frac{1}{2j} = \frac{1}{2}(-j)$$

$$a_{7} = -\frac{1}{2j} = \frac{1}{2}j$$

Each sinusoid of $x(t)$ produces a peak in the amplitude Fourier spectrum. There are two figures below, figure 48 showing magnitude and phase of Fourier coeffients and figure 49 showing the same with frequencies of this problem.

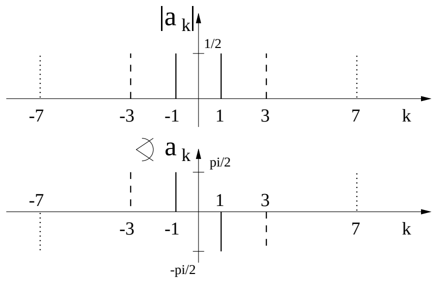


Figure 48: Fourier series representation of a continuous-time signal $x(t)$

Sampling.

The signal is sampled with sampling frequency $f_s = 1/T$.

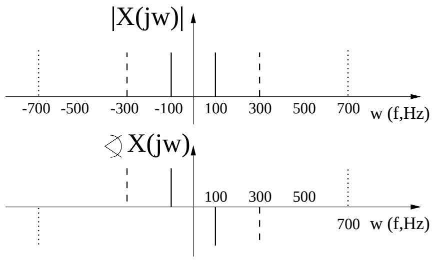


Figure 49: Same as in the figure above but now magnitude and phase representation of a continuous-time signal $x(t)$

$$x[n] = x(nT) = x(\frac{n}{f_s}) = \left(sin(2\pi\frac{f_1}{f_s}n) + sin(2\pi\frac{f_2}{f_s}n) + sin(2\pi\frac{f_3}{f_s}n) \right)$$

The sampling process can be seen as a multiplication of the continuous-time signal $g_a(t)$ by a periodic impulse train $p(t)$ (eq. 5.5, Mitra p. 285). An alternative form of the continuous-time Fourier transform of a sampled $g_p(t)$ is given by copying the Fourier transform of $g_a(t)$ (which is $G_a(j\omega)$) (eq. 5.9 Mitra p. 286)

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T))$$

See the figure 50. Because $|G_a|$ is symmetric, copying can be also seen as flipping around Nyquist frequency ($\Omega_{T/2}$). Remember that $sin(-\omega) = -sin(\omega)$ and $cos(-\omega) = cos(\omega)$. So, the phase of sine changes $\pi = 180$ degrees when flipping.

When reconstruating signals, we can only observe frequencies up to Nyquist frequency. If there are frequencies over Nyquist frequency in the original signal, those frequencies occur aliasing.

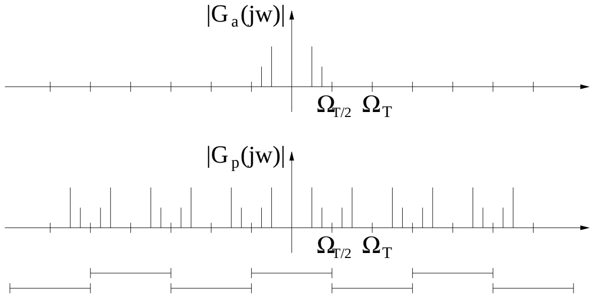


Figure 50: $G_a(j\Omega)$ and its sampled $G_p(j\Omega)$

Exercise examples.

In this problem $X(j\omega)$ is sampled with three different sampling frequencies f_s of 1500 Hz, 800 Hz and 400 Hz, The Nyquist frequency is the half of the sampling frequency $f_s/2$, 750 Hz, 400 Hz, and 200 Hz, respectively. Let f_m be the biggest frequency found in the input signal.

In the following figures for i, ii and iii, the scale and magnitude values for aliased frequencies are not exactly correct. Different phases may cause that a pure addition of magnitudes is not true. See problem 2b.

i) $f_s = 1500Hz$, $f_s/2 = 750Hz > f_m$. There is no aliasing. All three frequencies can be recovered. Figure 51.

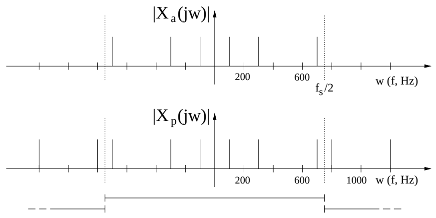


Figure 51: *Problem 50(b) i*

ii) $f_s = 800Hz$, $f_s/2 = 400Hz < f_m$. There is aliasing at frequency 100 Hz. In addition, the original frequency at 700 Hz is missed; it cannot be observed.

$$\begin{aligned} & sin(2\pi\frac{100}{800}n) + sin(2\pi\frac{300}{800}n) - sin(2\pi\frac{700}{800}n) = \\ &= sin(2\pi\frac{100}{800}n) + sin(2\pi\frac{300}{800}n) - sin(2\pi\frac{-100}{800}n) = \\ &= sin(2\pi\frac{100}{800}n) + sin(2\pi\frac{300}{800}n) + sin(2\pi\frac{100}{800}n) = \\ &= 2sin(2\pi\frac{100}{800}n) + sin(2\pi\frac{300}{800}n) \end{aligned}$$

See figure 52.

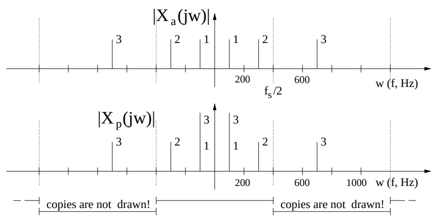


Figure 52: *Problem 50(b) ii*

iii) $f_s = 400Hz$, $f_s/2 = 200Hz < f_m$. There is aliasing at frequency 100 Hz. In addition, the original frequencies at 300 and 700 Hz are missed; they cannot be observe.

$$sin(2\pi\frac{100}{400}n) + sin(2\pi\frac{300}{400}n) - sin(2\pi\frac{700}{400}n)$$
$$= sin(2\pi\frac{100}{400}n)$$

See figure 53.

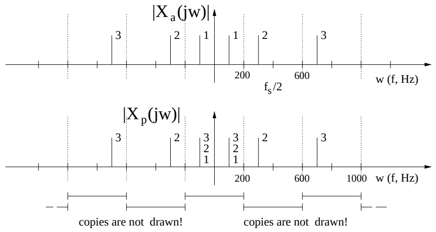


Figure 53: *Problem 50(b) iii*

c) From sampling theorem we know that a signal can be restored originally only if its highest frequency is less than half of the sampling frequency.
High frequency aliases into lower frequencies which can be observed. See the examples of Matlab demos `p5_01` and `p5_02` in the directory /p/edu/tik-61.140/ of Computer Centre workstations.

51. Signal reconstruction from samples

a) For a real signal $X(j\omega)$ is symmetric. For example figure 54.

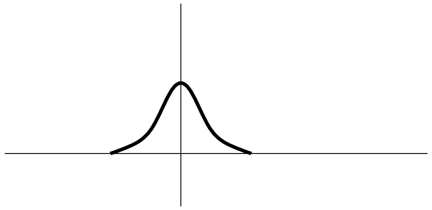


Figure 54: Arbitrary $X(j\omega)$

b) Discrete Fourier-transform is 2π-periodic ($2\pi/T$). For example, figure 55.

c) We can restore the orignal signal by lowpass filtering. See the figure 56.

d) When

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{elsewhere} \end{cases}$$

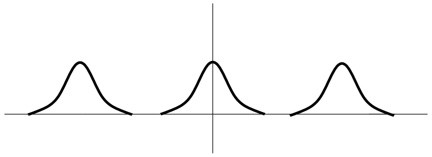


Figure 55: Continuous $X_p(j\omega)$

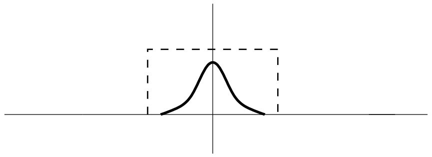


Figure 56: Reconstructed $X_r(j\omega)$

then by calculating or from the table

$$h(t) = \frac{sin(\omega_c t)}{\pi t} = \frac{\omega_c}{\pi} sinc(\omega_c t/\pi)$$

For each discrete measurement point can be thought to have a sinc-function. The sum of all these sincs will be the reconstructed signal. See the figure 7.10(c) from the book.

52. ...