Exercise 14.11.2006. Marginalization paradox[Jaynes94]

Assume we have n observations of independent, positive, exponentially distributed real values $\{x_1, \ldots, x_n\}$. First ζ values have expectations $\frac{1}{\eta}$, and the remaining $n - \zeta$ have expectations $\frac{1}{c\eta}$, where $c \neq 1$. Now c is known, and ζ, η unknown.

We are interested in the value of ζ.

1° Given the likelihood

$$p(x|\zeta, \eta) = c^{n-\zeta} \eta^n \exp\{-\eta \left(\sum_{i=1}^{\zeta} x_i + c \sum_{i=\zeta+1}^{n} x_i \right)\}, \quad 1 \leq \zeta \leq n,$$

and the prior $\pi(\zeta)\pi(\eta)$, form the marginal posterior $p(\zeta|x)$. Compute the marginal posterior by assuming an improper prior $\pi(\eta) = \eta^{-k}$.

2° Alternatively, one may inspect ratios $z_i = \frac{x_i}{x_1}$, since we can then get rid of a scaling parameter η. The likelihood is then

$$\begin{aligned} p(z_1, \ldots, z_n|\zeta, \eta) &= \int_0^\infty c^{n-\zeta} \eta^n \exp\{-\eta x_1 \left(\sum_{i=1}^{\zeta} z_i + c \sum_{i=\zeta+1}^{n} z_i \right)\} x_1^{n-1} dx_1 \\ &= \frac{c^{n-\zeta}(n-1)!}{(\sum_{i=1}^{\zeta} z_i + \sum_{i=\zeta+1}^{n} z_i)^n} = p(z_1, \ldots, z_n|\zeta). \end{aligned}$$

Form the posterior $p(\zeta|x)$, and compare the result to 1°.

Why do the two approaches give different results? When do the approaches agree? Does the situation change if you use proper priors?