Exercise for Bayes reading circle

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This exercise is from Gelman et al. (2nd ed.), page 136.
Consider a hierarchical normal model

$$\theta \sim N(\mu, \tau) \tag{1}$$
$$y_{ij} \sim N(\theta, \sigma_j). \tag{2}$$

with data y, and parameters μ, τ. Assume that $\{\sigma_j\}$ are known. Denote $\bar{y}_{.j} = \frac{1}{N_i} \sum_{i=1}^{N_i} y_{ij}$
The posterior distribution for τ is

$$p(\tau|y) = p(\tau) \frac{\prod_{j=1}^{J} N(\bar{y}_{.j}|\hat{\mu}, \sigma_j^2 + \tau^2)}{N(\hat{\mu}|\hat{\mu}, V)} \tag{3}$$
$$\sim p(\tau) V^{1/2} \prod_{j=1}^{J} (\sigma_j^2 + \tau^2)^{-1/2} exp(-\frac{(\bar{y}_{.j} - \hat{\mu})^2}{2(\sigma_j^2 + \tau^2)}), \tag{4}$$

where

$$\hat{\mu} = \frac{\sum_{j=1}^{J} \frac{1}{\sigma_j^2+\tau^2} \bar{y}_{.j}}{\sum_{j=1}^{J} \frac{1}{\sigma_j^2+\tau^2}} \tag{5}$$
$$V^{-1} = \sum_{j=1}^{J} \frac{1}{\sigma_j^2 + \tau^2} \tag{6}$$

The task is to show that the choice of prior $p(\tau) \sim 1$ leads to a proper posterior distribution. Morover, show that the seemingly non-informative prior $p(log(\tau)) \sim 1$ leads to improper posterior.