Chapter 4: Modelling

Exchangeability and Invariance

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Outline

Statistical models

- Events of interest defined in terms of random quantities $x_1, \dots, x_n$
- Individual's degrees of belief specified as a joint distribution function $P(x_1, \dots, x_n)$ (or density $p(x_1, \dots, x_n)$)
- In an application a specific form for p is chosen
- Here we study more general belief structures that lead to a mathematical representation of a model

Exchangeability

- Sometimes the indices of the random quantities $x_1, \ldots, x_n$ are judged not to be significant
- This leads to the notion of finite exchangeability

Definition (Finite exchangeability)

The random quantities $x_1, \ldots, x_n$ are finitely exchangeable under a probability measure P if

$$P(x_1, \ldots, x_n) = P(x_{\pi(1)}, \ldots, x_{\pi(n)})$$

for all permutations π.

- Partial and infinite exchangeability

The Bernoulli model

- Infinite exchangeability for 0-1 random quantities $\Longrightarrow$ the Bernoulli model

Theorem (Representation theorem for 0-1 random quantities)

If $x_1, x_2, \ldots$ are 0-1 random quantities and infinitely exchangeable under P, their joint mass function p is of the form

$$p(x_1, \ldots, x_n) = \int_0^1 \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} dQ(\theta),$$

where $Q(\theta) = \lim_{n\to\infty} P(y_n/n \le \theta)$ with $y_n = x_1 + \cdots + x_n$.

The multinomial model

- Infinite exchangeability for 0-1 random vectors $\implies$ the multinomial model

Theorem

If $\mathbf{x}_1, \mathbf{x}_2, \ldots$ are 0-1 random vectors and infinitely exchangeable under P, their joint mass function p is of the form

$$p(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \int_\Theta \prod_{i=1}^{n} \theta_1^{x_{i1}} \ldots \theta_k^{x_{ik}} \left(1 - \sum_{j=1}^{k} \theta_j\right)^{1 - \sum_j x_{ij}} dQ(\boldsymbol{\theta}),$$

where $Q(\boldsymbol{\theta}) = \lim_{n\to\infty} P((\bar{x}_{1n} \leq \theta_1) \cup \cdots \cup (\bar{x}_{kn} \leq \theta_k))$, with $\bar{x}_{in} = n^{-1}(x_{1i} + \cdots + x_{ni})$.

The general model

- Infinite exchangeability for real-valued random quantities $\Longrightarrow$ something rather abstract:

Theorem

If $x_1, x_2, \ldots$ are real-valued infinitely exchangeable random quantities under probability measure P, the form of P is

$$P(x_1, \ldots, x_n) = \int_{\mathcal{F}} \prod_{i=1}^{n} F(x_i) dQ(F),$$

where $Q(F) = \lim_{n\to\infty} P(F_n)$, F_n being the empirical distribution defined by $x_1, \ldots, x_n$.

Spherical symmetry

- The general representation theorem does not provide a concrete usable model
- More assumptions needed in addition to infinite exchangeability

Definition (Spherical symmetry)

A random vector $\mathbf{x} = [x_1, \ldots, x_n]$ has spherical symmetry under P, if $P(\mathbf{x}) = P(\mathbf{Ax})$ for all orthogonal matrices $\mathbf{A}$.

The normal model

- Spherical symmetry $\Longrightarrow$ the normal model

Theorem (Representation theorem under spherical symmetry)

If $x_1, x_2, \ldots$ is an infinite sequence of real-valued random quantities with probability measure P, and if, for any n, $\mathbf{x}_n := [x_1, \ldots, x_n]$ has spherical symmetry, the distribution of $\mathbf{x}_n$ has the form

$$P(\mathbf{x}_n) = \int_0^\infty \prod_{i=1}^{n} \Phi(\lambda^{1/2} x_i) dQ(\lambda),$$

where Φ is the standard normal distribution function and $Q(\lambda) = \lim_{n\to\infty} P(s_n^{-2} \leq \lambda)$ with $s_n^2 := n^{-1}(x_1^2 + \cdots + x_n^2)$.

Origin invariance

- Exchangeability of positive random quantities $\implies$ symmetry of events w.r.t. the 45° line through the origin
- Extension of this:

Definition (Origin invariance)

An infinitely exchangeable sequence $x_1, x_2, \ldots$ of positive real-valued random quantities with probability measure P has origin invariance if for all n and any event $A \subset \mathbb{R}_+^n$

$$P((x_1, \ldots, x_n) \in A) = P((x_1, \ldots, x_n) \in A + \mathbf{a})$$

for all $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a}^T\mathbf{1} = 0$.

The exponential model

- Origin invariance $\Longrightarrow$ the exponential model

Theorem (Continuous representation under origin invariance)

If the sequence $x_1, x_2, \ldots$ of positive real-valued random quantities has origin invariance under P, the joint density has the form

$$p(x_1, \ldots, x_n) = \int_0^\infty \prod_{i=1}^n \theta \exp(-\theta x_i) dQ(\theta)\,,$$

where $Q(\theta) = \lim_{n\to\infty} P(\bar{x}_n^{-1} \leq \theta)$ with $\bar{x}_n = n^{-1}(x_1 + \cdots + x_n)$.

Exercise

Come up or find in literature a model for a sequence of real-valued random variables $x_1, x_2, \ldots$ that is infinitely exchangeable and whose representation in terms of the general representation theorem genuinely involves an integral over measures. Write down the model using the notation of Proposition 4.3.